Measure, category and projective wellorders

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We show that each admissible assignment of $\aleph_1$ and $\aleph_2$ to the cardinal invariants in the Cichón Diagram is consistent with the existence of a projective wellorder of the reals.

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1 Introduction

Every admissible assignment of $\aleph_1$-$\aleph_2$ to the cardinal invariants in the Cichón diagram can be realized in a generic extension of a model of CH obtained as the countable support iteration of proper forcing notions (see [2, Chapter 7]). With every invariant in the Cichon diagram, one can associate a forcing notion which increases its value without affecting the values of the other invariants. Thus to a certain extent the problem of realizing an $\aleph_1$-$\aleph_2$ assignment in a generic extension, reduces to iterating certain posets (controlling the corresponding invariants) without introducing undesirable reals.

In [3] the first two authors provide a generic extension of the constructible universe L, in which there is a Δ^1_3-definable wellorder on the reals and $\mathfrak{c} = \aleph_2$. The extension is obtained by a countable support iteration of S-proper posets for some fixed stationary $S \subseteq \omega_1$. The construction leaves enough space to control in addition some of the combinatorial cardinal invariants of the continuum and it is established that each of the inequalities $\mathfrak{d} < \mathfrak{c}$, $\mathfrak{b} < \mathfrak{s} = \mathfrak{a}$, $\mathfrak{b} < \mathfrak{g}$ is consistent with the existence of a projective wellorder on $\mathbb{R}$. In the present paper we use the flexibility of this construction to control the invariants of measure and category. We show that each admissible $\aleph_1$-$\aleph_2$ assignment of the invariants in the Cichón diagram is consistent with the existence of a Δ^1_3 definable wellorder of the reals. In addition, in two instances we use a slight modification of the method from [4] which produces a Δ^1_3 definable wellorder of the reals via a finite support iteration of σ-centered posets.

The poset which forces the definable wellorder of the reals and is introduced in [3], can be presented in the form $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ where $\mathbb{Q}_\alpha = \mathbb{Q}^0_\alpha * \dot{\mathbb{Q}}^1_\alpha$ is a two step iteration: an arbitrary S-proper poset of size at most $\aleph_1$, for some stationary $S \subseteq \omega_1$ chosen in advance, followed be a three step iteration $\mathbb{Q}^1_\alpha$ of the form $\mathbb{K}^0_\alpha * \dot{\mathbb{K}}^1_\alpha * \dot{\mathbb{K}}^2_\alpha$. The poset $\mathbb{K}^0_\alpha$ shoots closed unbounded sets through certain components of a countable sequence of stationary sets (see [3, Definition 3]), $\mathbb{K}^1_\alpha$ is a poset known as localization (see [3, Definition 1]), and $\mathbb{K}^2_\alpha$ is a poset known as coding with perfect threes (see [3, Definition 3]). The poset $Q(T)$ of shooting a club through the stationary, co-stationary set T is $\omega_1\backslash T$-proper and ω-distributive. The localization poset $\mathcal{L}(\phi)$ is proper and does not add new reals. The only poset of these three forcing notions which does add a real is the coding with perfect trees partial order, denoted $\mathcal{C}(Y)$. $\mathcal{C}(Y)$ is proper and as we shall see below has the Sacks property.

Thus the main task in merging the above techniques is to show that the coding with perfect trees poset has all relevant iterable combinatorial properties and that in all relevant preservation theorems the requirement of properness can be relaxed to S-properness. In the applications of the finite support iteration forcing techniques which produce a wellorder of the reals, it is also of importance that the values of the relevant invariants can be controlled using σ-centered posets.

The paper is organized as follows: in section 2 we establish the necessary preservation theorems for S-proper, rather than proper iterations, in section 3 we study the combinatorial properties of $\mathcal{C}(Y)$ and in section 4 we show that each of the above admissible assignments is consistent with the existence of a Δ^1_3-w.o. on $\mathbb{R}$.

2 Preservation theorems

Throughout this section S denotes a stationary subset of ω_1.

For $T \subseteq \omega_1$ a stationary, co-stationary set let $Q(T)$ denote the poset of all countable closed subsets of $\omega_1\backslash T$ with extension relation given by end-extension. Note that if G is a $Q(T)$-generic set, then $\bigcup G$ is a closed unbounded subset of ω_1 which is disjoint from T. Thus $Q(T)$ destroys the stationarity of T. One of the main properties of $Q(T)$ which will be used throughout the paper is the fact that $Q(T)$ is ω-distributive and so does not add new reals (see [8]).

Since $Q(T)$ destroys the stationarity of T, it is not proper. However $Q(T)$ is $\omega_1\backslash T$-proper.

Definition 2.1 Let $T \subseteq \omega_1$ be a stationary set. A poset $\mathbb{Q}$ is T-proper, if for every countable elementary submodel $\mathcal{M}$ of $H(\Theta)$, where Θ is a sufficiently large cardinal, such that $\mathcal{M}\cap\omega_1\in T$, every condition $p\in\mathbb{Q}\cap\mathcal{M}$ has an $(\mathcal{M},\mathbb{Q})$-generic extension q.

The proofs of the following two statements can be found in [5].

Lemma 2.2 *If $\mathbb{Q}$ is S-proper, then $\mathbb{Q}$ preserves ω_1. Also $\mathbb{Q}$ preserves the stationarity of every stationary subset S' of ω_1 which is contained in S.*

Lemma 2.3 *If $\langle\langle\mathbb{P}_\alpha : \alpha\leq\delta\rangle,\langle\hat{\mathbb{Q}}_\alpha : \alpha<\delta\rangle\rangle$ is a countable support iteration of S-proper posets, then $\mathbb{P}_\delta$ is S-proper.*

The proofs of the following two statements follow very closely the corresponding “proper forcing iteration" case (see [1, Theorem 2.10 and 2.12]).

Lemma 2.4 *Assume CH. Let $\langle\mathbb{P}_\alpha : \alpha\leq\delta\rangle$ be a countable support iteration of length $\delta\leq\omega_2$ of S-proper posets of size ω_1. Then $\mathbb{P}_\delta$ is $\aleph_2$-c.c.*

Lemma 2.5 *Assume CH. Let $\langle\mathbb{P}_\alpha : \alpha\leq\delta\rangle$ be a countable support iteration of length $\delta<\omega_2$ of S-proper posets of size ω_1. Then CH holds in $V^{\mathbb{P}_\delta}$.*

Preserving $V\cap 2^\omega$ as a dominating or as an unbounded family: A forcing notion $\mathbb{P}$ is said to be *${}^\omega\omega$-bounding* if the ground model reals $V\cap{}^\omega\omega$ form a dominating family in $V^{\mathbb{P}}$. This property is preserved under countable support iteration of proper forcing notions. A forcing notion $\mathbb{P}$ is said to be *weakly bounding* if the ground model reals $V\cap{}^\omega\omega$ form an unbounded family in $V^{\mathbb{P}}$. In contrast to the ${}^\omega\omega$-bounding property, this property of weak unboundedness is not preserved under countable support iterations of proper posets. There are well-known examples of two step iterations of weakly bounding posets, which add a dominating real over V. An intermediate property, which preserves the ground model reals as an unbounded family in countable support iterations of proper posets, is the almost ${}^\omega\omega$-boundedness. A forcing notion $\mathbb{P}$ is said to be *almost ${}^\omega\omega$-bounding* if for every $\mathbb{P}$-name for a real $\dot{f}$, i.e. a $\mathbb{P}$-name for a function in ${}^\omega\omega$, and for every condition $p\in\mathbb{P}$, there is a real $g\in{}^\omega\omega\cap V$ such that for every $A\in[\omega]^\omega\cap V$ there is an extension $q\leq p$ such that $q\Vdash\exists^\infty i\in\check{A}(\dot{f}(i)\leq\check{g}(i))$. These are our main tools in providing that the ground model reals remain a dominating and or an unbounded family in the various models which we are to consider in section 4.

The proofs of the two preservation theorems below follow very closely the proofs of the classical preservation theorems concerning preservation of the ${}^\omega\omega$-bounding and the almost ${}^\omega\omega$-bounding properties respectively under countable support iterations of proper forcing notions (see [1], or [5]).

Lemma 2.6 *Let $\langle\langle \mathbb{P}_i : i \leq \delta\rangle, \langle\dot{\mathbb{Q}}_i : i < \delta\rangle\rangle$ be a countable support iteration of length $\delta \leq \omega_2$ of S-proper, ${}^\omega\omega$-bounding posets. That is, assume that for all $i < \delta$, $\Vdash_{\mathbb{P}_i}$ "$\dot{\mathbb{Q}}_i$ is ${}^\omega\omega$-bounding and S-proper". Then $\mathbb{P}_\delta$ is ${}^\omega\omega$-bounding and S-proper.*

Lemma 2.7 *Let $\langle\langle \mathbb{P}_i : i \leq \delta\rangle, \langle\dot{\mathbb{Q}}_i : i < \delta\rangle\rangle$ be a countable support iteration of length $\delta \leq \omega_2$ of S-proper, almost ${}^\omega\omega$-bounding posets. That is, assume that for all $i < \delta$,*

$$\Vdash_{\mathbb{P}_i} \text{"}\dot{\mathbb{Q}}_i \text{ is almost } {}^\omega\omega\text{-bounding and } S\text{-proper"}.$$

Then $\mathbb{P}_\delta$ is weakly bounding and S-proper.

Keeping non($\mathcal{M}$), non($\mathcal{N}$) *and* cof($\mathcal{N}$) *small:* Recall that with every ideal $\mathcal{I}$ on a set X we can associate the following invariants:

- $\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} \notin \mathcal{I}\}$,
- $\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} = X\}$,
- $\mathrm{non}(\mathcal{I}) = \min\{|Y| : Y \subseteq X \text{ and } Y \notin \mathcal{I}\}$, and
- $\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \forall B \in \mathcal{I} \exists A \in \mathcal{A}(B \subseteq A)\}$.

Following standard notation we denote by $\mathcal{M}$ and $\mathcal{N}$ the ideals of meager and null subsets of the real line, respectively. Thus add($\mathcal{M}$), cov($\mathcal{M}$), non($\mathcal{M}$), cof($\mathcal{M}$) and add($\mathcal{N}$), cov($\mathcal{N}$), non($\mathcal{N}$), cof($\mathcal{N}$) denote the above defined cardinal invariants for the ideals $\mathcal{M}$ and $\mathcal{N}$.

To preserve small witnesses to non($\mathcal{M}$), non($\mathcal{N}$) and cof($\mathcal{N}$) we will use preservation theorems which follow the general framework developed by M. Goldstern in [6].

Definition 2.8 ([2, Definition 6.1.6]) Let $\sqsubseteq$ be the union of an increasing sequence $\langle \sqsubseteq_n \rangle_{n\in\omega}$ of two place relations on ${}^\omega\omega$ such that

- the sets $\mathbb{C} = \mathrm{dom}(\sqsubseteq)$ and $\{f \in {}^\omega\omega : f \sqsubseteq_n g\}$ where $n \in \omega$, $g \in {}^\omega\omega$ are closed and have absolute definitions, that is, as Borel sets they have the same Borel codes in all transitive models.
- $\forall A \in [\mathbb{C}]^{\leq\aleph_0} \exists g \in {}^\omega\omega \forall f \in A(f \sqsubseteq g)$.

Let N be a countable elementary submodel of $H(\Theta)$ for some sufficiently large Θ containing $\sqsubseteq$. We say that $g \in {}^\omega\omega$ *covers* N if $\forall f \in N \cap \mathbb{C}(f \sqsubseteq g)$.

Following [2, Definition 6.1.7], we say that the poset $\mathbb{P}$ *S-almost-preserves-$\sqsubseteq$* if the following holds: whenever N is a countable elementary submodel of $H(\Theta)$ for some sufficiently large Θ containing $\mathbb{P}$, $\mathbb{C}$, $\sqsubseteq$ and $\omega_1 \cap N \in S$, g covers N and $p \in \mathbb{P} \cap N$, then there is an $(N, \mathbb{P})$-generic condition q extending p such that $q \Vdash$ "g covers $N[\dot{G}]$". Similarly we say that the forcing notion $\mathbb{P}$ *S-preserves-$\sqsubseteq$* if $\mathbb{P}$ satisfies [2, Definition 6.1.10] with respect only to countable elementary submodels whose intersection with ω_1 is an element of the stationary set S. More precisely, we say that a forcing notion $\mathbb{P}$ *S-preserves-$\sqsubseteq$* if whenever N is a countable elementary submodel of $H(\Theta)$ for some sufficiently large Θ which contains $\mathbb{P}$ and $\sqsubseteq$ as elements and such that $\omega_1 \cap N \in S$, whenever g covers N and $\langle p_n \rangle_{n\in\omega}$ is a sequence of conditions interpreting the $\mathbb{P}$-names $\langle \dot{f}_i \rangle_{i \leq k} \in N$ for functions in $\mathbb{C}$ as the functions $\langle f_i^* \rangle_{i \leq k}$, then there is an N-generic condition $q \leq p_0$ such that $q \Vdash_{\mathbb{P}}$ "g covers $N[\dot{G}]$" and

$$\forall n \in \omega \forall i \leq k \; q \Vdash_{\mathbb{P}} (f_i^* \sqsubseteq_n g \to \dot{f}_i \sqsubseteq_n g).$$

Furthermore we obtain the following analogue of Goldstern's preservation theorem (see [6] or [2, Theorem 6.1.3]).

Theorem 2.9 *Let S be a stationary set and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ be a countable support iteration such that for all $\alpha < \delta$, $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ S-preserves-$\sqsubseteq$". Then $\mathbb{P}_\delta$ S-preserves-$\sqsubseteq$.*

Of particular interest for us are the relations $\sqsubseteq^{random}$, $\sqsubseteq^{Cohen}$ and $\sqsubseteq^{\Delta}$ defined in Definitions 6.3.7, 6.3.15 of [2] and on page 303 of [2], respectively. For convenience of the reader we define these relations below:

$\sqsubseteq^{random}$: Denote by Ω the set of all clopen subsets of 2^ω. Then let

$$\mathbb{C}^{\text{random}} = \{f \in \Omega^\omega : \forall n \in \omega(\mu(f(n)) \leq 2^{-n})\}$$

and for $f \in \mathbb{C}^{\text{random}}$ let $A_f = \bigcap_{n\in\omega} \bigcup_{k \geq n} f(k)$. Now for $f \in \mathbb{C}^{\text{random}}$, $x \in 2^\omega$ and $n \in \omega$ define

$$f \sqsubseteq_n^{\text{random}} x \iff \forall k \geq n (x \notin f(k)).$$

Let $\sqsubseteq^{random} = \bigcup_{n\in\omega} \sqsubseteq_n^{random}$. Note that $f \sqsubseteq^{\text{random}} x$ if and only if $x \notin A_f$ and that x covers N with respect to $\sqsubseteq^{\text{random}}$ if and only if x is random over N.

$\sqsubseteq^{Cohen}$: Let

$$\mathbb{C}^{\text{Cohen}} = \{f \in \Omega^\Omega : \forall U \in \Omega(f(U) \subseteq U)\}.$$

For $f \in \mathbb{C}^{\text{Cohen}}$ let $A_f := \bigcup_{U\in\Omega} f(U)$. Note that A_f is an open dense subset of 2^ω and that for every dense open set $H \subseteq 2^\omega$ there is an $f \in \mathbb{C}^{\text{Cohen}}$ such that $A_f \subseteq H$. Fix some standard enumeration $\{U_n\}_{n\in\omega}$ of Ω and for $f \in \mathbb{C}^{\text{Cohen}}$, $x \in 2^\omega$, $n \in \omega$ define:

$$f \sqsubseteq_n^{\text{Cohen}} x \iff \exists k \leq n (x \in f(U_k)).$$

Let $\sqsubseteq^{Cohen} = \bigcup_{n\in\omega} \sqsubseteq_n^{Cohen}$. Then $f \sqsubseteq^{Cohen} x$ if and only if $x \in A_f$. Therefore x covers N with respect to $\sqsubseteq^{Cohen}$ if and only if x is a Cohen real over N.

$\sqsubseteq^{\Delta}$: Let $\mathbb{Q}_+ = \mathbb{Q} \cap [0,1]$, let $\Delta = \{f \in \mathbb{Q}_+^{\omega} : \sum_{n\in\omega} f(n) < 1\}$ and let

$$\mathbb{C}^{\Delta} := \{f \in ((\mathbb{Q}_+)^{<\omega})^{\omega} : \forall n \sum_{i\in \mathrm{dom}(f(n))} f(n)(i) < 2^{-(n+1)}\}.$$

For $f \in \mathbb{C}^{\Delta}$ let $\epsilon_f \in \Delta$ be defined by $\epsilon_f = f(0)^\frown f(1)^\frown \cdots$. For $f, g \in \mathbb{C}^{\Delta}$ define

$$f \sqsubseteq_n^{\Delta} g \iff \forall m \geq n(\epsilon_f(m) \leq \epsilon_g(m)).$$

Let $\sqsubseteq^{\Delta} = \bigcup_{n\in\omega} \sqsubseteq_n^{\Delta}$.

Each of those relations satisfies the properties of Definition 2.8. Thus Theorem 2.9 implies the following two theorems (analogous to Theorems 6.1.13 and 6.3.20 from [2] respectively).

Theorem 2.10 *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta\rangle$ is a countable support iteration and for each $\alpha < \delta$, $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ S-preserves- $\sqsubseteq^{random}$ ", then $\mathbb{P}_\delta$ preserves outer measure. That is for every set $A \subseteq 2^\omega$, $V^{\mathbb{P}_\delta} \models \mu^*(A) = \mu^*(A)^V$. In particular $\Vdash_\delta V \cap 2^\omega \notin \mathcal{N}$.*

Theorem 2.11 *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta\rangle$ is a countable support iteration and for each $\alpha < \delta$, $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ S-preserves- $\sqsubseteq^{Cohen}$ ", then $\mathbb{P}_\delta$ preserves non meager sets. That is for every set $A \subseteq 2^\omega$ which is not meager, $V^{\mathbb{P}_\delta} \models A$ is not meager. In particular $\Vdash_\delta V \cap 2^\omega \notin \mathcal{M}$.*

Recall that a forcing notion $\mathbb{P}$ has the *Sacks property* if and only if for every $\mathbb{P}$-name $\dot{g}$ for a function in ${}^\omega\omega$ there is a slalom $S \in V$, i.e. a function $S \in ([\omega]^{<\omega})^\omega$ such that $|S(n)| \leq 2^n$ for all n, such that $\Vdash_{\mathbb{P}}$ "$\forall n(\dot{g}(n) \in S(n))$". By [2, Lemma 6.3.39] a proper forcing notion $\mathbb{P}$ is has the Sacks property if and only if $\mathbb{P}$ preserves $\sqsubseteq^{\Delta}$. By [2, Theorem 2.3.12] if $\mathbb{P}$ has the Sacks property then every measure zero set in $V^{\mathbb{P}}$ is covered by a Borel measure zero set in V and so $\mathbb{P}$ preserves the base of the ideal of measure zero sets. We obtain the following analogue of [2, Theorem 6.3.40].

Theorem 2.12 *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta\rangle$ is a countable support iteration and for each $\alpha < \delta$, $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ S-preserves- $\sqsubseteq^{\Delta}$ ", then $\mathbb{P}_\delta$ has the Sacks property and so preserves the base of the ideal of measure zero sets.*

No random and no amoeba reals: Some of the preservation theorems which we use to show that certain iterations do not add amoeba or random reals, are based on a general framework due to Judah and Repický (see [7]).

Definition 2.13 ([2, Definition 6.1.17]) Let $\sqsubseteq$ be the union of an increasing chain $\langle \sqsubseteq_n \rangle_{n\in\omega}$ of two place relations on ${}^\omega\omega$ such that

- for all $n \in \omega$ and all $h \in {}^\omega\omega$ the set $\{x : h \sqsubseteq_n x\}$ is relatively closed in the range of $\sqsubseteq$,
- for every $A \in [\mathrm{dom}(\sqsubseteq)]^{\leq\aleph_0}$ there is $f \in \mathrm{dom}(\sqsubseteq)$ such that $\forall g \in A \forall n \in \omega \exists k \geq n$ such that $\forall x(f \sqsubseteq_k x) \to g \sqsubseteq_k x)$, and
- the formula $\forall x \in {}^\omega\omega(f \sqsubseteq_n x \to g \sqsubseteq_n x)$ is absolute for all transitive models containing f and g.

A real x is said to be $\sqsubseteq$-*dominating* over V if for all $y \in V \cap \mathrm{dom}(\sqsubseteq)$, $y \sqsubseteq x$.

We have the following S-proper analogue of Judah and Repický's preservation theorem (see [2, Theorem 6.1.18]).

Theorem 2.14 *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta\rangle$, δ limit, is a countable support iteration of S-proper posets, such that for all $\alpha < \delta$, $\mathbb{P}_\alpha$ does not add a $\sqsubseteq$-dominating real, then $\mathbb{P}_\delta$ does not add a $\sqsubseteq$-dominating real.*

Note that $x \in 2^\omega$ $\sqsubseteq^{\text{random}}$-dominates V if and only if x is random over V. Furthermore the relation $\sqsubseteq^{\text{random}}$ satisfies the conditions of definition 2.13 and so by the above theorem we obtain the following S-proper analogue of Theorem 6.3.14 from [2].

Theorem 2.15 *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta\rangle$, δ limit, is a countable support iteration of S-proper forcing notions and for each $\alpha < \delta$, $\mathbb{P}_\alpha$ does not add random reals, then $\mathbb{P}_\delta$ does not add a random real.*

Note that $\sqsubseteq^\Delta$ also satisfies the conditions of Definition 2.13. Then by Theorem 2.14 above, as well as [2, Theorem 2.3.12] we obtain the following analogue of [2, Theorem 6.3.41].

Theorem 2.16 *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta\rangle$, δ limit, is a countable support iteration of S-proper posets and for all $\alpha < \delta$, $\Vdash_\alpha$ "$\bigcup(\mathcal{N} \cap V) \notin \mathcal{N}$", then $\Vdash_\delta \bigcup(\mathcal{N} \cap V) \notin \mathcal{N}$".*

Other preservation theorems: We say that a forcing notion $\mathbb{P}$ is S-(f,h)-*bounding*, if it satisfies [2, Definition 7.2.13] but instead of proper we require that $\mathbb{P}$ is S-proper. That is, we say that $\mathbb{P}$ *is* S-(f,h)-*bounding*, if $\mathbb{P}$ is S-proper, for every $k \in \omega$ $\lim_{n\to\infty} h(n)^k \cdot f^{-1}(n) = 0$ and for every $f' \in V^{\mathbb{P}} \cap \prod_{n\in\omega} f(n)$ there is $S \in V \cap ([\omega]^{<\omega})^\omega$ such that for all $n \in \omega$ $|S(n)| \leq h(n)$ and for all $n \in \omega(f'(n) \in S(n))$. The proof of [2, Lemma 7.2.15] remains true under this modification, and so we obtain that if $\mathbb{P}$ is S-(f,h)-bounding then $\mathbb{P}$ does not add random or Cohen reals. Furthermore we have the following analogue of Shelah's theorem (see [9] or see [2, Theorem 7.2.19]).

Theorem 2.17 *If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$, δ limit, is a countable support iteration such that for all α, $\Vdash_\alpha$ "$\dot{\mathbb{Q}}_\alpha$ is S-(f,h)-bounding", then $\mathbb{P}_\delta$ is S-(f,h)-bounding.*

We will also use preservation theorems for the so called (F,g)-preserving posets. For convenience of the reader we state the definition of (F,g)-preserving (see [2, Definition 7.2.23]). Let g be a given real and for $n \in \omega$ let $P_n = \{a \subseteq g(n+1) : |a| = g(n+1)/2^n\}$. For a set $A \subseteq P_n$ define $\mathrm{norm}(A) = \min\{|X| : \forall a \in A (X \not\subseteq a)\}$. Let F be a family of strictly increasing functions. For every $f \in F$ choose a function $f^+ \in F$ and assume that for all $f \in F$, $n \in \omega$ we have that $f(n) < g(n)/2^n$. A forcing notion $\mathbb{P}$ is said to be (F,g)-*preserving* if for every $f \in F$ and every $\mathbb{P}$-name $\dot{S}$ which has the property that for all n, $\Vdash_{\mathbb{P}} \dot{S}(n) \subseteq P_n$ and $\Vdash_{\mathbb{P}} \mathrm{norm}(\dot{S}(n)) < f(n)$, there exists a function $T \in V$ such that for all n, $T(n) \subseteq P_n$, $\mathrm{norm}(T(n)) < f^+(n)$ and $\Vdash_{\mathbb{P}} \dot{S}(n) \subseteq T(n)$. Note that the countable support iteration of (F,g)-preserving posets is (F,g)-preserving (see [2, Theorem 7.2.29]) and that (F,g)-preserving posets do not add Cohen reals (see [2, Theorem 7.2.24]).

3 Coding with perfect trees

Let $Y \subseteq \omega_1$ be generic over L such that in $L[Y]$ cofinalities have not been changed and let $\bar{\mu} = \{\mu_i\}_{i \in \omega_1}$ be a sequence of L-countable ordinals such that μ_i is the least ordinal μ with $\mu > \bigcup\{\mu_j : j < i\}$, $L_\mu[Y \cap i] \vDash \mathrm{ZF}^-$ and $L_\mu \vDash \omega$ is the largest cardinal. A real R is said to *code* Y below i if for all $j < i$, $j \in Y$ if and only if $L_{\mu_j}[Y \cap j, R] \vDash \mathrm{ZF}^-$. Whenever T is a perfect tree, let $|T|$ be the least i such that $T \in L_{\mu_i}[Y \cap i]$.

Fix $L[Y]$ as the ground model. The poset $\mathcal{C}(Y)$, to which we refer as coding with perfect trees, consists of all perfect trees T such that every branch R through T codes Y below $|T|$. For T_0, T_1 conditions in $\mathcal{C}(T)$ define $T_0 \leq T_1$ if and only if T_0 is a subtree of T_1.

Below we systemize some of the main properties of the poset $\mathcal{C}(Y)$. Note that $T_0 \leq T_1$ if and only if $[T_0] \subseteq [T_1]$ where $[T]$ denotes the set of infinite branches through T. For $n \in \omega$, let $T_0 \leq_n T_1$ if and only if $T_0 \leq T_1$ and T_0, T_1 have the same first n splitting levels. For T a perfect tree, $m \in \omega$ let $S_m(T)$ be the set of nodes on the m-splitting level of T (and so $|S_m(T)| = 2^m$), and for $t \in T$ let $T(i) = \{\eta \in T : t \subseteq \eta \text{ or } \eta \subseteq t\}$. Note that by absoluteness R codes Y below $|T|$ even for branches through T in the generic extension.

Lemma 3.1 *[3, Lemma 5] If $T \in \mathcal{C}(Y)$ and $|T| \leq i < \omega_1$, then there is $T^* \leq T$ such that $|T^*| = i$.*

Lemma 3.2 *[3, Lemma 6] If G is $\mathcal{C}(Y)$-generic and $\{R\} = \bigcap\{[T] : T \in G\}$, then for all $j < \omega_1$ we have that*

$$j \in Y \text{ if and only if } L_{\mu_j}[Y \cap j, R] \models ZF^-.$$

That is, R codes Y.

Lemma 3.3 *[3, Lemmas 7 and 8] $\mathcal{C}(Y)$ is a proper, ${}^\omega\omega$-bounding forcing notion.*

Lemma 3.4 *The coding with perfect trees forcing notion $\mathcal{C}(Y)$ preserves $\sqsubseteq^{Cohen}$.*

Proof Let N be a countable elementary submodel of $L_\Theta[Y]$ for some sufficiently large Θ, such that $\mathcal{C}(Y)$, $\bar{\mu}$ are elements of N. Let c be a Cohen real over N. Let T be a condition in $\mathcal{C}(Y) \cap N$. We have to show that there is a condition T^* which is a $(N, \mathcal{C}(Y))$-generic extension of T and which forces that "c is Cohen over $N[\dot{G}]$". By [2, Lemma 2.2.4] for every meager set $F \subseteq 2^\omega$ there are reals $x_F \in 2^\omega$ and $f_F \in \omega^\omega$ such that

$$F \subseteq \{x : \forall^\infty n \exists i \in [f_F(n), f_F(n+1)) x_F(i) \neq x(i)\}.$$

Let $\{\dot{x}_n, \dot{f}_n\}_{n\in\omega}$ and $\{D_n\}_{n\in\omega}$ enumerate names for representatives of all meager sets in $N^{\mathcal{C}(Y)}$ and all dense subsets of $\mathcal{C}(Y)$ in N, respectively. Let $\overline{N}$ denote the transitive collapse of N, let $i = \omega_1 \cap N$. Note that $\overline{N} = L_\mu[Y \cap i]$ for some μ and since $L_{\mu_i}[Y \cap i] \models$ "i is countable", we have that $L_\mu[Y \cap i]$ is an element of $L_{\mu_i}[Y \cap i]$. Let $\bar{\imath} = \{i_k\}_{k\in\omega}$ be an increasing cofinal sequence in i such that $\bar{\imath} \in L_{\mu_i}[Y \cap i]$. Recursively we will define a sequence of conditions $\tau = \{T_n\}_{n\in\omega}$, such that for every n, the condition T_n is an element of N, $T_{n+1} \leq_{n+1} T_n$, $|T_n| \geq i_n$ and

(1) $T_{2n} \Vdash_{\mathcal{C}(Y)}$ "$c \notin F(\dot{x}_n, \dot{f}_n)$", where $F(\dot{x}_n, \dot{f}_n)$ denotes a name for the meager set corresponding to the names $\dot{x}_n, \dot{f}_n$,

(2) $T_{2n+1} \Vdash_{\mathcal{C}(Y)}$ "$\dot{G} \cap N \cap D_n \neq \emptyset$", where $\dot{G}$ is the canonical $\mathcal{C}(Y)$-name for the generic filter.

Furthermore the entire sequence τ will be an element of $L_{\mu_i}[Y \cap i]$, since it will be definable in $L_{\mu_i}[Y \cap i]$. Thus its fusion T^* will also be an element of $L_{\mu_i}[Y \cap i]$, and so a condition in $\mathcal{C}(Y)$ which extends T and has the desired properties.

We will need the following two claims:

Claim *Let $R \in \mathcal{C}(Y) \cap N$ and let $\{\dot{x}, \dot{f}\}$ be $\mathcal{C}(Y)$-names in N (for reals), representing a meager set in $N^{\mathcal{C}(Y)}$, let $n \in \omega$ and let $\alpha \in N \cap \omega_1$ such that $\alpha > |R|$. Then there is a condition R' in N such that $R' \leq_n R$, $|R'| \geq \alpha$ and every branch through R' decides $\dot{x}, \dot{f}$.*

Proof Let N_0 be a sufficiently elementary submodel of N such that $N \models$ "N_0 is countable" and all relevant parameters are elements of N_0, that is R, $\mathcal{C}(Y)$, $\bar{\mu}$, $\dot{f}$, $\dot{x}$, n and α are elements of N_0. Let $\overline{N_0}$ denote the transitive collapse of N_0 and let $j = \omega_1 \cap N_0$. Note that $\overline{N}_0$ is of the form $L_\mu[Y \cap j]$ for some μ and since $L_\mu[Y \cap j] \models$ "j is uncountable" and $L_{\mu_j}[Y \cap j] \models$ "j is countable", we have that $\overline{N}_0 = L_\mu[Y \cap j] \in L_{\mu_j}[Y \cap j]$. On the other hand, since $L_{\mu_j}[Y \cap j]$ is definable from Y, j, μ_j and all of those are in N, we obtain that $L_{\mu_j}[Y \cap j] \in N$. Let $\bar{j} = \{j_m\}_{m \in \omega}$ be an increasing cofinal in j sequence, which is an element of $L_{\mu_j}[Y \cap j]$.

The condition R' will be obtained as the fusion of a sequence $\langle R_m \rangle_{m \in \omega}$ such that the entire sequence is definable in $L_{\mu_j}[Y \cap j]$ and for all m, $R_m \in N_0$ (and so $R_m \in \overline{N}_0$). Let $R_0 = R$. For every $s \in \mathrm{Split}_n(R_0)$ and every $t \in \mathrm{Succ}_s(R_0)$ find $R^0_t \leq R_0(t)$ which decides $\dot{x}\restriction|t|$ and $\dot{f}\restriction|t|$. By elementarity, we can assume that $R^0_t \in N_0$ and so $R^0_t \in \bar{N}_0$. Since the set of conditions in $\mathcal{C}(Y)$ of height strictly greater than α and j_0 is dense, again by elementarity we can assume that $|R^0_t| > \alpha, j_0$. Let $R_1 = \bigcup_{s \in \mathrm{Split}_n(R_0)} \bigcup_{t \in \mathrm{Succ}_t(R_0)} R^0_t$. Then in particular $R_1 \in N_0$ and $|R_1| > \alpha, i_0$. Now suppose $R_m \in N_0$ is defined. Then for every $s \in \mathrm{Split}_{n+m}(R_m)$ and $t \in \mathrm{Succ}_s(R_m)$ find $R^m_t \leq R_m(t)$ in $\bar{N}_0$ of height $> \alpha, j_m$, which decides $\dot{x}\restriction|t|$, $\dot{f}\restriction|t|$. Let $R_{m+1} = \bigcup_{s \in \mathrm{Split}_{n+m}(R_m)} \bigcup_{t \in \mathrm{Succ}(s)} R^m_t$. Then $R_{m+1} \leq_{m+n} R_m$, $R_{m+1} \in N_0$ and $|R_{m+1}| > \alpha, j_m$. With this the inductive construction of the fusion sequence is complete. Since $\langle R_m \rangle_{m \in \omega}$ is definable in $L_{\mu_j}[Y \cap j]$, we obtain that $R' = \bigcap_{m \in \omega} R_m \in L_{\mu_j}[Y \cap j]$. Then in particular $|R'| = j$, which implies that R' is indeed a condition in $\mathcal{C}(Y)$. □

Claim *Let R', $\dot{x}$, $\dot{f}$, n, α, N be as above and let c be a Cohen real over N. Then there is a condition $R'' \in N$ such that $R'' \leq_n R'$, $|R''| \geq \alpha, |R'|$ and R'' forces that c does not belong to the meager set determined by $\dot{x}, \dot{f}$.*

Proof Just as in the previous claim let N_0 be a sufficiently elementary submodel of N such that $N \models$ "N_0 is countable" and all relevant parameters are elements of N_0. Let $\overline{N_0}$ denote the transitive collapse of N_0. Let $j = \omega_1 \cap N_0$ and let $\bar{j} = \{j_m\}_{m \in \omega}$ be an increasing and cofinal in j sequence which is an element of $L_{\mu_j}[Y \cap j]$. The condition R'' will be obtained as the limit of a fusion sequence $\langle R_m \rangle_{m \in \omega}$ which is definable in $L_{\mu_j}[Y \cap j]$ and whose elements are in N_0. Let $R_0 = R'$. For every $s \in \mathrm{Split}_n(R_0)$ and every $t \in \mathrm{Succ}_t(R_0)$ find a branch $b_t \in N_0 \cap [R_0]$ such that $t \subseteq b_t$. Then b_t gives an interpretation of the names $\dot{x}$, $\dot{f}$ as reals x^t and f^t in N_0. Since c is Cohen over N, it is Cohen over N_0 and so there is $j_t > |t|$ such that

$$x^t_n\restriction[f^t(j_t), f^t(j_t+1)) = c\restriction[f^t(j_t), f^t(j_t+1)).$$

Take any $k_t > j_t$. Let $R_1 = \bigcup_{s\in\mathrm{Split}_n(R_0)}\bigcup_{t\in\mathrm{Succ}_s(R_0)} R_0(b_t\restriction k_t)$. Thinning out once again we can assume that $|R_0(b_t\restriction k_t)| > j_0, \alpha$. Also clearly $R_1 \in N_0$.

Suppose R_m is defined. Again for every $s \in \mathrm{Split}_{n+m}(R_m)$ and $t \in \mathrm{Succ}_s(R_m)$, find a branch $b_t \in [R_m]\cap N_0$ such that $t \subseteq b_t$. Then b_t gives an interpretation x^t, f^t of $\dot{x}, \dot{f}$ as reals x^t, f^t in N_0. Using the fact that c is Cohen over N_0 we can find $\{l^t_a\}_{1\leq a\leq m}$ such that $|t| < l^t_1$, $l^t_a < l^t_{a+1}$ for $a < m$ such that for every $j \in \{l^t_a\}_{1\leq a\leq m}$

$$x^t\restriction[f^t(j), f^t(j+1)) = c\restriction[f^t(j), f^t(j+1)).$$

Take any $k_t > l^t_m$. Let $R_{m+1} = \bigcup_{s\in\mathrm{Split}_{n+m}(R_m)}\bigcup_{t\in\mathrm{Succ}_s(R_m)} R_m(b_t\restriction k_t)$. Passing to an extension if necessary we can assume that $|R_m(b_t\restriction k_t)| > j_m, \alpha$ and so that $|R_{m+1}| > j_m, \alpha$. Let $R'' = \cap_{m\in\omega} R_m$. Then R'' is a condition in N with the desired properties. $\square$

With this we can proceed with the construction of the fusion sequence $\langle T_n\rangle_{n\in\omega}$. Let $T_0 = T$. Reproducing the proof of [3, Lemma 7] find $T_1 \in N$ such that $T_1 \leq_1 T_0$, $|T_1| \geq i_1$ and $T_1 \Vdash \dot{G}\cap N\cap D_1 \neq \emptyset$. Suppose T_{2n-1} is defined for some $n \geq 1$. Using the previous two claims find a condition $T_{2n} \in N\cap\mathcal{C}(Y)$ such that $|T_{2n}| \geq i_{2n}$, $T_{2n} \leq_{2n} T_{2n-1}$, and T_{2n} forces that c does not belong to the meager set corresponding to $\{\dot{x}_n, \dot{f}_n\}$. Obtain T_{2n+1} as in the base case. With this the fusion sequence $\langle T_n\rangle_{n\in\omega}$ is defined. Let $T^* = \bigcap_{n\in\omega} T_n$. Note that $|T^*| = i$ and so in particular $T \in \mathcal{C}(Y)$. Clearly, T^* is $(N, \mathcal{C}(Y))$-generic and $T^* \Vdash_{\mathcal{C}(Y)}$ "c is Cohen over $N[\dot{G}]$.". $\square$

In order to show that the coding with perfect tress forcing notion, preserves $\sqsubseteq^{\text{random}}$, we will use the fact that $\mathcal{C}(Y)$ is weakly bounding and that $\mathcal{C}(Y)$ preserves positive outer measure (see below).

Lemma 3.5 *Suppose that A is a set of positive outer measure. Then $\Vdash_{\mathcal{C}(Y)} \mu^*(A) > 0$.*

Proof Suppose not. Then there is a condition $T \in \mathcal{C}(Y)$ such that $T \Vdash \mu^*(A) = 0$. Let N be a countable elementary submodel of $L_\Theta[Y]$ for some sufficiently large Θ such that $T, \mathcal{C}(Y), A$ are elements of N. Then there is a sequence $\langle \dot{I}_n\rangle_{n\in\omega} \in N$ of rational intervals such that $T \Vdash \sum_{n\in\omega}\mu(\dot{I}_n) < \infty$ and $T \Vdash A \subseteq \bigcap_{n\in\omega}\bigcup_{m\geq n}\dot{I}_m$. Then in particular, there is a $\mathcal{C}(Y)$-name for a function $\dot{g}$ in ${}^\omega\omega$ such that for all n, $T \Vdash \sum_{m\geq\dot{g}(n)}\mu(\dot{I}_m) < 2^{-(n^2+n)}$. Then there is $R \leq T$ and a ground model real g, i.e. function in ${}^\omega\omega$ such that for all $n \in \omega$, $R \Vdash \dot{g}(n) < \check{g}(n)$. Then in particular, for all $n \in \omega$, $R \Vdash \sum_{g(n)\leq i<g(n+1)}\mu(\dot{I}_i) < 2^{-(n^2+n)}$. Let $i = \omega_1\cap N$ and let $\bar{\imath} = \{i_n\}_{n\in\omega}$ be an increasing and cofinal in i sequence, which belongs to $L_{\mu_i}[Y\cap i]$. Recursively define a fusion sequence $\langle R_n\rangle_{n\in\omega}$ as follows. Let $R_0 = R$. Suppose R_n has been defined. For every n-splitting node t of R_n find $R_t \leq R_n(t)$ such that for some finite sequence $\langle I^n_{t,j}\rangle_{g(n)\leq j<g(n+1)}$ of rational intervals for all $j : g(n) \leq j < g(n+1)$ we have $R_t \Vdash \dot{I}_j = \check{I}^n_{t,j}$. By elementarity we can assume that R_t is a condition which is an element of N, which is also of height $\geq i_n$ and that $\langle I^n_{t,j}\rangle_{g(n)\leq j<g(n+1)} \in N$. Let $R_{n+1} = \bigcup_{t\in \mathrm{Split}_n(R_n)} R_t$ and let $J_n = \bigcup_{t\in \mathrm{Split}_n(R_n)} \bigcup_{g(n)\leq j<g(n+1)} I^n_{t,j}$. Note that $J_n \in N$ and $\mu(J_n) < 2^{-n}$. Let R^* be the fusion of the sequence $\langle R_n\rangle_{n\in\omega}$. Then R^* is a condition in $\mathcal{C}(Y)$ of height i, such that

$$R^* \Vdash \bigcap_n \bigcup_{m\geq n} \dot{I}_m \subseteq \bigcap_n \bigcup_{m\geq n} J_m.$$

Since $J := \bigcap_n \bigcup_{m\geq n} J_m$ is a measure zero set, there is $x \in A\backslash J$. However

$$R^* \Vdash x \in \bigcap_n \bigcup_{m\geq n} \dot{I}_m$$

and so $R^* \Vdash x \in J$, which is a contradiction. □

Lemma 3.6 *The coding with perfect trees forcing notion $\mathcal{C}(Y)$ preserves $\sqsubseteq^{\mathrm{random}}$.*

Proof The proof proceeds similarly to the proof that Laver forcing preserves $\sqsubseteq^{\mathrm{random}}$ (see [2, Theorem 7.3.39]). Let N be a countable elementary submodel of $L_\Theta[Y]$ for some sufficiently large Θ, let $\dot{f}_0$ be an element of $\dot{\mathbb{C}}^{\mathrm{random}} \cap N$ and let $\tau = \langle T_n\rangle_{n\in\omega} \in N$ be an approximating sequence for $\dot{f}_0$ below T for some $T \in \mathcal{C}(Y)\cap N$. Let f^*_0 be the approximation of $\dot{f}_0$ determined by τ. Note that $f^*_0 \in N \cap {}^\omega\Omega$. Let x be a random real over N. We have to show that there is an extension T^* of T which is an $(N, \mathcal{C}(Y))$-generic condition, such that $T^* \Vdash$ "x is random over $N[\dot{G}]$" and such that for all $n \in \omega$, $T^* \Vdash (f^*_0 \sqsubseteq_n x \to \dot{f}_0 \sqsubseteq_n x)$.

Let D be a dense open subset of $\mathcal{C}(Y)$. Denote by $\mathrm{cl}(D) = \{T : \exists n\forall t \in \mathrm{Split}_{\geq n}(T)(\text{if there is } R_t \leq_0 T(t) \text{ such that } R_t \in D \text{ then } T(t) \in D)\}$. Note that $\mathrm{cl}(D)$ is n-dense open for every $n \in \omega$ and so if $\{D_n\}_{n\in\omega}$ is a sequence of dense open sets, then $\bigcap_{n\in\omega} \mathrm{cl}(D_n)$ is n-dense for all n. Also, we have that if $S \leq T \in \mathrm{cl}(D)$, then there is $s \in S$ such that $T(s) \in D$.

Let $\mathcal{D}$ denote the collection of all dense subsets of $\mathcal{C}(Y)$ which are in N. Since x is random over N and $f^*_0 \in N$ there is n_0 such that for all $k \geq n_0(x \notin f^*_0(k)$. For every $n \geq n_0$ let Y^n_n be the set of all reals $z \in 2^\omega$ such that there is $Z \leq T_n$ such that $\phi_n(z, Z)$ holds where $\phi_n(z, Z)$ is the conjunction of the following three formulas:

(1) $\phi_1(Z) \equiv$ for all $D \in \mathcal{D} \cap N \exists R \in \mathrm{cl}(D) \cap N(Z \leq R)$, and

(2) $\phi_2(z, T) \equiv$ for all $\dot{f} \in \dot{\mathbb{C}}^{\mathrm{random}} \cap N \forall^\infty n(Z \Vdash z \notin \dot{f}(n))$,

(3) $\phi_3^n(z,T) \equiv$ for all $k \geq n$ $Z \Vdash z \notin \dot{f}_0(k)$.

Note that $Z \not\Vdash z \notin \dot{f}(n)$ iff there is $Z' \leq Z$ such that $Z' \Vdash z \in \dot{f}(n)$ iff there is $Z' \leq Z$ such that $z \in \dot{f}(n)[Z']$ which is equivalent to there is $s \in Z$ such that $z \in \dot{f}(n)[Z_s]$ iff there is $R \in \mathrm{cl}(D_n^{\dot{f}}) \cap N \exists s \in Z(Z \geq R$ and $z \in \dot{f}(n)[R_s])$. Since quantifiers of ϕ_1, ϕ_2, ϕ_3 are relativized to subsets of N, all three of these formulas are Borel.

Using the fact that $\mathcal{C}(Y)$ is weakly homogeneous and preserves positive outer measure, modifying the proof of [2, Lemma 7.3.41] we obtain that for every $n \geq n_0$, the inner measure $\mu_*(Y_n^n) \geq 1 - 2^{-n}$. This implies that $Y^* := \bigcup_{n \geq n_0} Y_n^n$ is a set of measure 1.

Claim *(see [2, Lemma 7.3.42]) There is a sequence $\langle B_k : k \geq n_0 \rangle \in N$ of Borel sets such that for all n, $B_n \in N$ and $B_n \triangle Y_n^n \subseteq \bigcup(\mathcal{N} \cap N)$.*

Proof Fix $z \in 2^\omega$ and let G be an $N[z]$-generic filter for $\mathrm{Coll}(2^{2^{\aleph_0}}, \aleph_0)$ (the algebra collapsing $2^{2^{\aleph_0}}$ onto $\aleph_0$). Now we have $z \in Y_n^n$ iff $L_\Theta[Y] \models \exists Z \leq T \phi_n(z,Z)$ iff $N[z][G] \models \exists Z \leq T \phi_n(z,T)$ iff $N[z] \Vdash$ " $\Vdash_{\mathrm{Coll}(2^{2^{\aleph_0}}, \aleph_0)} \exists Z \leq T \phi_n(z,Z)$". The second equivalence follows from absoluteness of Σ_1^1 formulas and the third from homogeneity of $\mathrm{Coll}(2^{2^{\aleph_0}}, \aleph_0)$.

Let $\phi_n^*(z)$ denote the formula " $\Vdash_{\mathrm{Coll}(2^{2^{\aleph_0}}, \aleph_0)} \exists Z \leq T \phi_n(z,Z)$". That is $z \in Y$ iff $N[z] \models \phi_n^*(z)$. Let B_n be a Borel set in N representing the Boolean value $[\![\phi_n^*(\dot{r})]\!]_{\mathbb{B}}$ where $\dot{r}$ is the canonical name for a random real. For a random real z over N we have,

$$z \in Y_n^n \iff N[z] \models \phi_n^*(z) \iff z \in B_n.$$

Therefore $B_n \triangle Y_n^n \subseteq \bigcup(\mathcal{N} \cap N)$. □

Note that in particular $\mu(B_n) \geq 1 - 2^{-n}$. Using the fact that x is random over N we obtain that there is $n^* \geq n_0$ such that $x \in B_{n^*}$. Again since $B_{n^*} \triangle Y_{n^*}^{n^*} \subseteq \bigcup(\mathcal{N} \cap N)$, $x \in Y_{n^*}^{n^*}$. Let T^* be a witness to $x \in Y_{n^*}^{n^*}$. Then $T^* \leq T_{n^*}$, T^* is $(N, \mathcal{C}(Y))$-generic, $T^* \Vdash$ "x is random over N" and for all $k \geq n^*$, $T^* \Vdash x \notin \dot{f}_0(k)$. Then

$$T^* \Vdash f_0^* \restriction n^* = \dot{f}_0 \restriction n^* \wedge \forall k \geq n(x \notin \dot{f}_0(k))$$

which implies that for all $n \in \omega$, $T^* \Vdash (f_0^* \sqsubseteq_n x \to \dot{f}_0 \sqsubseteq_n x)$. □

Recall that a forcing notion $\mathbb{P}$ has the *Laver property* if and only if for every function $f \in V \cap \omega^\omega$ and a $\mathbb{P}$-name $\dot{g}$ such that $\Vdash_{\mathbb{P}} \forall n(\dot{g}(n) \leq f(n))$ there is a slalom $S \in V$ such that $\Vdash_{\mathbb{P}} \forall n \dot{g}(n) \in S(n)$.

Lemma 3.7 *Sacks coding $\mathcal{C}(Y)$ has the property L_f where $f(n) = 2^n$ for all n, and so has the Laver property. It is ${}^\omega\omega$-bounding and so has the Sacks property. Furthermore it is (F, g)-preserving for some F and g (see [2, Definition 7.2.23]) and is (f, h)-bounding for all f and h.*

Proof Recall that a forcing notion $\mathbb{P}$ has property L_f where $f \in {}^\omega\omega$, if for every $p \in \mathbb{P}$, $n \in \omega$ and $A \in [\omega]^{<\omega}$ the following holds: if $p \Vdash \dot{a} \in A$, then there is $q \leq_n p$ and $B \subseteq A$, $|B| \leq f(n)$ such that $q \Vdash \dot{a} \in B$. Thus suppose $T \in \mathcal{C}(Y)$, $n \in \omega$ and $A \in [\omega]^{<\omega}$ such that $T \Vdash \dot{a} \in \check{A}$. Let $S_n(T)$ be the n-th splitting level of T. Then $|S_n(T)| = 2^n$ and for every $t_j \in S_n(T)$ there is $T'_j \leq T(t_j)$ such that $T'_j \Vdash \dot{a} = \check{k}_j$ for some $k_j \in A$. Let $B = \{k_j\}_{j \in 2^n} \subseteq A$, $T' = \bigcup_{j \in 2^n} T'_j$. Then $T' \leq_n T$ and $T' \Vdash \dot{a} \in \check{B}$. By [2, Lemma 7.2.2] if $\mathbb{P}$ has the L_f property for some f, then $\mathbb{P}$ has the Laver property. Since $\mathcal{C}$ is ${}^\omega\omega$-bounding, by [2, Lemma 6.3.38] it has the Sacks property. The Laver property implies also that $\mathcal{C}(Y)$ is (F, g)-preserving for some F and g (see [2, Lemma 7.2.25] and is (f, h)-bounding for all f and h (see [2, Lemma 7.2.16]). □

4 Measure, category and projective wellorders

The underlying forcing construction is the construction from [3] forcing a Δ^1_3-w.o. of the reals. For completeness of the argument we will give a brief outline of this construction. Recall that a transitive ZF^- model M is *suitable* if ω_2^M exists and $\omega_2^M = \omega_2^{L^M}$. Assume V is the constructible universe L. Let $F : \omega_2 \to L_{\omega_2}$ be a bookkeeping function which is Σ_1-definable over L_{ω_2} and let $\bar{S} = (S_\beta : \beta < \omega_2)$ be a sequence of almost disjoint stationary subsets of ω_1 which is Σ_1-definable over L_{ω_2} with parameter ω_1, such that $F^{-1}(a)$ is unbounded in ω_2 for every $a \in L_{\omega_2}$, and whenever M, N are suitable models such that $\omega_1^M = \omega_1^N$ then $F^M, \bar{S}^M$ agree with $F^N, \bar{S}^N$ on $\omega_2^M \cap \omega_2^N$. In addition if M is suitable and $\omega_1^M = \omega_1$, then $F^M, \bar{S}^M$ equal the restrictions of F,$\bar{S}$ to the ω_2 of M. Let S be a stationary subset of ω_1 which is Δ_1-definable over L_{ω_1} and almost disjoint from every element of $\bar{S}$.

Recursively define a countable support iteration $\langle\langle \mathbb{P}_\alpha : \alpha \leq \omega_2\rangle, \langle\dot{\mathbb{Q}}_\alpha : \alpha < \omega_2\rangle\rangle$ such that $\mathbb{P} = \mathbb{P}_{\omega_2}$ will be a poset adding a Δ^1_3-definable wellorder of the reals. We can assume that all names for reals are nice in the sense of [3] and that for $\alpha < \beta < \omega_2$ all $\mathbb{P}_\alpha$-names for reals precede in the canonical wellorder $<_L$ of L all $\mathbb{P}_\beta$-names for reals, which are not $\mathbb{P}_\alpha$-names. For each $\alpha < \omega_2$ define $<_\alpha$ as in [3]: that is if x, y are reals in $L[G_\alpha]$ and $\sigma_x^\alpha, \sigma_y^\alpha$ are the $<_L$-least $\mathbb{P}_\gamma$-names for x, y respectively, where $\gamma \leq \alpha$, define $x <_\alpha y$ if and only if $\sigma_x^\alpha <_L \sigma_y^\alpha$. Note that $<_\alpha$ is an initial segment of $<_\beta$. If

G is a $\mathbb{P}$-generic filter, then $<^G = \bigcup\{<^G_\alpha : \alpha < \omega_2\}$ will be the desired wellorder of the reals.

In the recursive definition of $\mathbb{P}_{\omega_2}$, $\mathbb{P}_0$ is defined to be the trivial poset and $\dot{\mathbb{Q}}_\alpha$ is of the form $\dot{\mathbb{Q}}^0_\alpha * \dot{\mathbb{Q}}^1_\alpha$, where $\dot{\mathbb{Q}}^0_\alpha$ is an arbitrary $\mathbb{P}_\alpha$-name for a proper forcing notion of cardinality at most $\aleph_1$ and $\dot{\mathbb{Q}}^1_\alpha$ is defined as in [3], and so carries out the task of forcing the Δ^1_3-w.o. of the reals. Note that $\mathbb{Q}^1_\alpha$ is the iteration of countably many posets shooting clubs through certain stationary, co-stationary sets from $\bar{S}$ (and so each of those is S-proper and ω-distributive), followed by a "localization" forcing which is proper and does not add new reals, followed by coding with perfect trees. In the following we will use the fact that $\dot{\mathbb{Q}}^0_\alpha$ is arbitrary, to force the various $\aleph_1$-$\aleph_2$-admissible assignments to the cardinal characteristics of the Cichón diagram, in the presence of a Δ^1_3 wellorder of the reals.

Theorem 4.1 *The constellation determined by* $\mathrm{cov}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \aleph_2$ *and* $\mathfrak{b} = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof Perform the countable support iteration described above, which forces a Δ^1_3-w.o. of the reals and in addition specify $\dot{\mathbb{Q}}^0_\alpha$ as follows. If α is even let $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = \mathbb{B}$ be the random real forcing, and if α is odd let $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = \mathbb{C}$ be the Cohen forcing. Then in $V^{\mathbb{P}_{\omega_2}}$ $\mathrm{cov}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \aleph_2$. At the same time since the countable support iteration of S-proper, almost ${}^\omega\omega$-bounding posets is weakly bounding, the ground model reals remain an unbounded family and so a witness to $\mathfrak{b} = \aleph_1$. □

Theorem 4.2 *The constellation determined by* $\mathfrak{d} = \aleph_2$, $\mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof In the forcing construction described above, which forces a Δ^1_3-w.o. of the reals, define $\dot{\mathbb{Q}}^0_\alpha$ to be the rational perfect tree forcing PT defined in [2, Definition 7.3.43]. To claim that $\mathfrak{d} = \aleph_2$ in the final generic extension, note that PT adds an unbounded real. It remains to show that $\mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \aleph_1$. By [2, Theorem 7.3.46] the rational perfect tree forcing preserves $\sqsubseteq^{\text{Cohen}}$ and by Lemma 3.4 the coding with perfect tress $\mathcal{C}(Y)$ also preserves $\sqsubseteq^{\text{Cohen}}$. Therefore by Theorem 2.11 in $V^{\mathbb{P}_{\omega_2}}$ the set $2^\omega \cap V$ is non meager and so $V^{\mathbb{P}_{\omega_2}} \Vdash \mathrm{non}(\mathcal{M}) = \aleph_1$. By [2, Theorem 7.3.47] the rational perfect tree forcing preserves $\sqsubseteq^{\text{random}}$ and by Lemma 3.6 the prefect tree coding $\mathcal{C}(Y)$ preserves $\sqsubseteq^{\text{random}}$. Therefore by Theorem 2.10 in the final extension $2^\omega \cap V$ is a non null set and so $V^{\mathbb{P}_{\omega_2}} \models \mathrm{non}(\mathcal{N}) = \aleph_1$. □

Theorem 4.3 *The constellation determined by* $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \aleph_2$, $\mathfrak{b} = \mathrm{cov}(\mathcal{M}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For even α, let $\dot{\mathbb{Q}}^0_\alpha$ be the random real forcing $\mathbb{B}$ and for α odd, let $\dot{\mathbb{Q}}^1_\alpha$ be the Blass-Shelah forcing notion $\mathbb{Q}$ defined in [2, 7.4.D]. Since all iterands are almost ${}^\omega\omega$-bounding, by Lemma 2.7 the ground model reals remain an unbounded family and so a witness to $\mathfrak{b} = \aleph_1$. On the other hand $\mathbb{Q}$ adds an unbounded real and $\Vdash_{\mathbb{Q}}$ "$2^\omega \cap V \in \mathcal{N}$", which implies that $V^{\mathbb{P}_{\omega_2}} \models \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \aleph_2$. Since cofinaly often we add random reals, we have that $\mathrm{cov}(\mathcal{N}) = \aleph_2$ in the final extension. To show that no Cohen reals are added by the iteration, use the fact that all iterands are (F, g)-preserving, as well as [2, Theorems 7.2.29 and 7.2.24]. □

Theorem 4.4 *The constellation determined by* $\mathrm{non}(\mathcal{M}) = \mathfrak{d} = \aleph_2$ *and* $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \mathrm{non}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}_\alpha = \mathrm{PT}_{f,g}$ and for α odd, let $\dot{\mathbb{Q}}_\alpha = \mathrm{PT}$ where $\mathrm{PT}_{f,g}$ and PT are defined in [2, Definition 7.3.43 and Definition 7.3.3] respectively. Since $\Vdash_{PT_{f,g}} 2^\omega \cap V \in \mathcal{M}$ and *PT* adds an unbounded real, $V^{\mathbb{P}_{\omega_2}} \models \mathrm{non}(\mathcal{M}) = \mathfrak{d} = \aleph_2$. All iterands are almost ${}^\omega\omega$-bounding and so $\mathfrak{b}$ remains small. All iterands S preserve $\sqsubseteq^{random}$ and so by Theorem 2.10 $\mathbb{P}_{\omega_2}$ preserves outer measure and so $V^{\mathbb{P}_{\omega_2}} \models \mathrm{non}(\mathcal{N}) = \aleph_1$. To see that the iteration does not add random reals, note that PT and $\mathcal{C}(Y)$ have the Laver property and so are (f, g)-bounding for all f, g. On the other hand $PT_{f,g}$ is (f, h)-bounding for some appropriate h, which implies that all iterands are S-(f, h)-bounding. Then by Theorem 2.17, $\mathbb{P}_{\omega_2}$ is S-(f, h)-bounding, which implies that is does not add random reals. □

Theorem 4.5 *The constellation determined by* $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_2$ *and* $\mathfrak{b} = \mathrm{non}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even, let $\dot{\mathbb{Q}}_\alpha$ be the rational perfect tree forcing PT and for α odd, let $\dot{\mathbb{Q}}_\alpha$ be the random real forcing $\mathbb{B}$. Then $V^{\mathbb{P}_{\omega_2}} \models \mathrm{cov}(\mathcal{N}) = \mathfrak{d} = 2^{\aleph_0}$. By [2, Theorem 6.3.12] $\mathbb{B}$ preserves $\sqsubseteq^{\mathrm{random}}$, by [2, Theorem 7.3.47] PT preserves $\sqsubseteq^{\mathrm{random}}$ and by Lemma 3.6 Sacks coding preserves $\sqsubseteq^{\mathrm{random}}$. Then Theorem 2.10, $V^{\mathbb{P}_{\omega_2}} \models 2^\omega \cap V \notin \mathcal{N}$. All iterands are almost ${}^\omega\omega$-bounding, and so by Theorem 2.7 the ground model reals remain an unbounded family in $V^{\mathbb{P}_{\omega_2}}$. □

Theorem 4.6 *The constellation determined by* $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \aleph_2$ *and* $\mathfrak{b} = \mathrm{cov}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even, let $\dot{\mathbb{Q}}^0_\alpha$ be Cohen forcing and for α odd, let $\dot{\mathbb{Q}}^0_\alpha$ be $\mathrm{PT}_{f,g}$ (see [2, Definition 7.3.3]). Since $\Vdash_{\mathrm{PT}_{f,g}} 2^\omega \cap V \in \mathcal{M}$, $V^{\mathbb{P}_{\omega_2}} \models \mathrm{non}(\mathcal{M}) = \aleph_2$. Since cofinally often we add Cohen reals, clearly $\mathrm{cov}(\mathcal{M}) = \aleph_2$ in the final generic extension. All involved partial orders are almost ${}^\omega\omega$-bounding and so $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{b} = \omega_1$. To see that the iteration does not add random reals, proceed by induction using Theorem 2.15 at limit steps.

Alternative Proof: The result can be obtained using finite support iteration of ccc posets. We will slightly modify the coding stage of the construction of [4]. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_2, \beta < \omega_2 \rangle$ be a finite support iteration such that $\mathbb{P}_0$ is the poset defined in [4, Lemma 1]. Suppose $\mathbb{P}_\alpha$ has been defined. If α is a limit, $\alpha = \omega_1 \cdot \alpha' + \xi$ where $\xi < \omega_1$ and $\alpha' > 0$, define $\mathbb{Q}_\alpha$ as in Case 1 of the original construction. If α is not of the above form, i.e. α is a successor or $\alpha < \omega_1$, let $\dot{\mathbb{Q}}_\alpha$ be a name for the following poset adding an eventually different real:

$$\mathbb{Q}_\alpha = \{\langle s_0, s_1 \rangle : s_0 \in \omega^{<\omega}, s_1 \in [\text{o.t.}(\dot{<}^G_\alpha)]^{<\omega}\},$$

where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ if and only if s_0 is an initial segment of t_0, $s_1 \subseteq t_1$, and for all $\xi \in s_1$ and all $j \in [|s_0|, |t_0|)$ we have $t_0(j) \neq x_\xi(j)$, where x_ξ is the ξ-th real in $L[G_\alpha] \cap \omega^\omega$ according to the wellorder $\dot{<}^{G_\alpha}_\alpha$. The sets $\dot{A}_\alpha$ are defined as in [4]. With this the definition of $\mathbb{P}_{\omega_2}$ is complete. Following the proof of the original construction one can show that $\mathbb{P}_{\omega_2}$ does add a Δ^1_3-definable wellorder on the reals (note that in our case $V^{\mathbb{P}_{\omega_2}} \vDash \mathfrak{c} = \aleph_2$.) Since the eventually different forcing adds a Cohen real and makes the ground model reals meager, we obtain that $V^{\mathbb{P}_{\omega_2}} \Vdash \mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) = \aleph_2$. Since all iterands of our construction are σ-centered, by [2, Theorems 6.5.30 and 6.5.29] $\mathbb{P}_{\omega_2}$ does not add random reals and so $V^{\mathbb{P}_{\omega_2}} \vDash \mathrm{cov}(\mathcal{N}) = \aleph_1$. The ground model reals remain an unbounded family and so a witness to $\mathfrak{b} = \aleph_1$ in $V^{\mathbb{P}_{\omega_2}}$. We should point out that the coding techniques allow one to obtain the consistency of the existence of a Δ^1_3 wellorder of the reals with $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \aleph_3$ and $\mathfrak{b} = \mathrm{cov}(\mathcal{N}) = \aleph_1$. □

Theorem 4.7 *The constellation determined by* $\mathfrak{d} = \mathrm{non}(\mathcal{N}) = \aleph_2$ *and* $\mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even, let $\dot{\mathbb{Q}}^0_\alpha$ be the rational perfect tree forcing PT and for α odd, let $\dot{\mathbb{Q}}^0_\alpha$ be the poset S_{g,g^*} (see [2, 7.3.C]). Note that $\Vdash_{S_{g,g^*}} 2^\omega \cap V \in \mathcal{N}$ and so $V^{\mathbb{P}_{\omega_2}} \vDash \mathrm{non}(\mathcal{N}) = \aleph_2$. On the other hand PT adds an unbounded real, which implies that $(\mathfrak{d} = \aleph_2)^{\mathbb{V}^{\mathbb{P}_{\omega_2}}}$. Also S_{g,g^*}, PT and $\mathcal{C}(Y)$ preserve $\sqsubseteq^{Cohen}$, which by Theorem 2.11 implies that $V^{\mathbb{P}_{\omega_2}} \vDash \mathrm{non}(\mathcal{M}) = \aleph_1$. To see that there are no Cohen reals added by the iteration we use the S-(f,g)-bounding property. More precisely, PT and $\mathcal{C}(Y)$ are Laver and so (f,g)-bounding for all f,g. The poset S_{g,g^*} is (g,g^*)-bounding, which implies that all iterands are S-(g,g^*)-bounding. Thus by Theorem 2.17, $\mathbb{P}_{\omega_2}$ is S-(g,g^*)-bounding, and so the entire iteration does not add Cohen reals. □

Theorem 4.8 *The constellation determined by* $\mathrm{cov}(\mathcal{M}) = \aleph_2$, $\mathrm{non}(\mathcal{M}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For every $\alpha < \omega_2$, let $\dot{\mathbb{Q}}^0_\alpha$ be Cohen forcing. By [2, Theorem 6.3.18] $\mathbb{C}$ preserves $\sqsubseteq^{\text{Cohen}}$, by Theorem 3.4 Sacks coding preserves $\sqsubseteq^{\text{Cohen}}$. Then by Theorem 2.11 the entire iteration $\mathbb{P}_{\omega_2}$ preserves non-meager sets and so in particular $V^{\mathbb{P}_{\omega_2}} \models 2^\omega \cap V \notin \mathcal{M}$. □

Theorem 4.9 *The constellation determined by* $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathrm{non}(\mathcal{M}) = \aleph_2$ *and* $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \mathrm{cov}(\mathcal{M}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α an even successor, let $\dot{\mathbb{Q}}^0_\alpha$ be the rational perfect tree forcing PT, for α an odd successor let $\dot{\mathbb{Q}}^0_\alpha$ be $\mathrm{PT}_{f,g}$ (see [2, Definition 7.3.3]) and for α limit, let $\dot{\mathbb{Q}}^0_\alpha = S_{g,g^*}$. Clearly $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \mathrm{non}(\mathcal{M}) = \aleph_2$. To show that $\mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \aleph_1$ use the fact that all forcing notions used in the iteration are S-(f,h)-bounding and so by Theorem 2.17 $\mathbb{P}_{\omega_2}$ is S-(f,h)-bounding. Thus no real in $V^{\mathbb{P}_{\omega_2}}$ is Cohen or random over V. To show that $\mathfrak{b} = \aleph_1$ in the final extension, use the facts that all iterands are almost ${}^\omega\omega$-bounding. □

Theorem 4.10 *The constellation determined by* $\mathrm{add}(\mathcal{N}) = \aleph_2$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof Note that if $\mathbb{A}$ is amoeba forcing then $V^{\mathbb{A}} \models \bigcup(\mathcal{N} \cap V) \in \mathcal{N}$. Thus in order to obtain the desired result, it is sufficient to require that for every every $\alpha < \omega_2$, $\dot{\mathbb{Q}}^0_\alpha$ is the amoeba forcing. □

Theorem 4.11 *The constellation determined by* $\mathrm{cof}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof Sacks coding has the Sacks property and so by [2, Lemma 6.3.39] $\mathcal{C}(Y)$ preserves $\sqsubseteq^\Delta$ (and so it S-preserves-$\sqsubseteq^\Delta$). For every α let $\dot{\mathbb{Q}}^\alpha_0$ be the trivial poset. Then by theorem 2.12 $\mathbb{P}_{\omega_2}$ preserves the base of the ideal of measure zero sets, that is $V^{\mathbb{P}_{\omega_2}} \models \mathrm{cof}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})^V = \aleph_1$. □

Theorem 4.12 *The constellation determined by* $\mathrm{add}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \aleph_2$ *and* $\mathrm{add}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even successor let $\dot{\mathbb{Q}}^0_\alpha$ be the random real forcing $\mathbb{B}$, for α odd successor let $\dot{\mathbb{Q}}^0_\alpha$ be Cohen forcing $\mathbb{C}$, and for α limit let $\dot{\mathbb{Q}}^0_\alpha$ be Laver forcing LT. Then clearly in $V^{\mathbb{P}_{\omega_2}}$ we have that $\mathrm{add}(\mathcal{M}) = \mathrm{cov}(\mathcal{N}) = \aleph_2$. To show that there are no amoeba reals in the final generic extension, and so $\mathrm{add}(\mathcal{N}) = \aleph_1$, proceed by induction using Theorem 2.16 at limit stages. □

Theorem 4.13 *The constellation determined by* $\mathrm{cof}(\mathcal{N}) = \aleph_2$ *and* $\mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{M}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For each α let $\mathbb{Q}^0_\alpha$ be the poset $\mathbb{U}$ defined on page 339 in [2]. This poset is ${}^\omega\omega$-bounding, preserves $\sqsubseteq^{random}$, preserves $\sqsubseteq^{Cohen}$ and does not have the Sacks property. By Theorem 2.6, the ground model reals dominate the reals in $V^{\mathbb{P}_{\omega_2}}$ and so $\mathfrak{d} = \aleph_1$. On the other hand since all iterands S-preserves-$\sqsubseteq^{random}$ and S-preserve-$\sqsubseteq^{Cohen}$, in $V^{\mathbb{P}_{\omega_2}}$ we have $\mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) = \aleph_1$. Thus in particular $\mathbb{V}^{\mathbb{P}_{\omega_2}} \vDash \mathrm{cof}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \aleph_1$. To see that $\mathrm{cof}(\mathcal{N}) = \aleph_2$ in $V^{\mathbb{P}_{\omega_2}}$ use the fact that $\mathbb{U}$ does not have the Sacks property. □

Theorem 4.14 *The constellation determined by* $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \mathrm{non}(\mathcal{N}) = \aleph_2$ *and* $\mathrm{cov}(\mathcal{M}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}^0_\alpha$ be random real forcing, for α an odd successor let $\dot{\mathbb{Q}}_\alpha$ be the poset S_{g,g^*} defined in [2, Section 7.3.C] and for α limit, let $\dot{\mathbb{Q}}^0_\alpha$ be Laver forcing. To see that $\mathrm{cov}(\mathcal{M}) = \aleph_1$ in the final generic extension, note that all iterands are (F, g)-preserving and so by [2, Theorems 7.2.29 and 7.2.24] $\mathbb{P}_{\omega_2}$ does not add Cohen reals. □

Theorem 4.15 *The constellation determined by the equations* $\mathrm{non}(\mathcal{M}) = \aleph_2$ *and* $\mathrm{non}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For each $\alpha < \omega_2$, let $\dot{\mathbb{Q}}^0_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathrm{PT}_{f,g}$. Note that by [2, Theorem 7.3.6] we have that $V^{\mathrm{PT}_{f,g}} \vDash V \cap {}^\omega\omega \in \mathcal{M}$. Therefore in $V^{\mathbb{P}_{\omega_2}}$ we have that $\mathrm{non}(\mathcal{M}) = \aleph_2$. The poset $\mathrm{PT}_{f,g}$ is (f, h)-bounding for some h, and so all iterands are S-(f, h)-bounding. Then by Theorem 2.17 $\mathbb{P}_{\omega_2}$ is S-(f, h)-bounding, which implies that $\mathbb{P}_{\omega_2}$ does not add random reals. Thus $\mathrm{cov}(\mathcal{N}) = \aleph_1$ in the final generic extension. Since all iterands are ${}^\omega\omega$-bounding, by Theorem 2.6 the ground model reals are a witness to $\mathfrak{d} = \omega_1$ in $V^{\mathbb{P}_{\omega_2}}$. By [2, Theorem 7.3.15] the poset $\mathrm{PT}_{f,g}$ preserves $\sqsubseteq^{\mathrm{random}}$, Sacks coding preserves $\sqsubseteq^{\mathrm{random}}$ and so by Theorem 2.10 $\mathbb{P}_{\omega_2}$ preserves outer measure. Thus $V^{\mathbb{P}_{\omega_2}} \vDash \mathrm{non}(\mathcal{N}) = \aleph_1$. □

Theorem 4.16 *The constellation determined by the equalities* $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \aleph_2$ *and* $\mathrm{non}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}^0_\alpha$ be the random real forcing $\mathbb{B}$ and for α odd let $\dot{\mathbb{Q}}^0_\alpha$ be Laver forcing LT. Then we immediately get that $\mathrm{cov}(\mathcal{N}) = \mathfrak{b} = \aleph_2$ in $V^{\mathbb{P}_{\omega_2}}$. By [2, Theorem 7.3.39] LT preserves $\sqsubseteq^{\mathrm{random}}$, by [2, Theorem 6.3.12] $\mathbb{B}$ preserves $\sqsubseteq^{\mathrm{random}}$ and Sacks coding preserves $\sqsubseteq^{\mathrm{random}}$. Then by Theorem 2.10 $V^{\mathbb{P}_{\omega_2}} \vDash 2^\omega \cap \mathcal{N} \notin \mathcal{N}$ and so $V^{\mathbb{P}_{\omega_2}} \vDash \mathrm{non}(\mathcal{N}) = \aleph_1$. □

Theorem 4.17 *The constellation determined by* $\mathrm{cov}(\mathcal{N}) = \aleph_2$ *and* $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For each α, let $\dot{\mathbb{Q}}^0_\alpha$ be the random real forcing $\mathbb{B}$. Since $\mathbb{B}$ and the Sacks coding preserve $\sqsubseteq^{\mathrm{random}}$, Theorem 2.10 implies that $V^{\mathbb{P}_{\omega_2}} \vDash \mathrm{non}(\mathcal{N}) = \aleph_1$. By Lemma 2.6 $\mathbb{P}_{\omega_2}$ is ${}^\omega\omega$-bounding and so $\mathfrak{d} = \aleph_1$ in the final generic extension. □

Theorem 4.18 *The constellation determined by* $\mathrm{add}(\mathcal{M}) = \aleph_2$ *and* $\mathrm{cov}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even, let $\dot{\mathbb{Q}}^0_\alpha$ be the Cohen forcing $\mathbb{C}$ and for α odd, let $\dot{\mathbb{Q}}^0_\alpha$ be the Laver forcing. Clearly $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\} = \aleph_2$ in $V^{\mathbb{P}_{\omega_2}}$. To show that $\mathbb{P}_{\omega_2}$ does not add random reals proceed by induction using Theorem 2.15 at limit steps.

Alternative proof: The result can be obtained using finite support iteration of ccc posets, by slightly modifying the coding stage of the poset forcing a Δ^1_3 definable wellorder on the reals from [4]. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta; \alpha \leq \omega_2, \beta < \omega_2 \rangle$ be a finite support iteration where $\mathbb{P}_0$ is the poset defined in [4, Lemma 1]. Suppose $\mathbb{P}_\alpha$ has been defined. If α is a limit and $\alpha = \omega_1 \cdot \alpha' + \xi$ where $\xi < \omega_1$ and $\alpha' > 0$, define $\mathbb{Q}_\alpha$ as in Case 1 of the original construction. Otherwise, if α is a successor or $\alpha < \omega_1$ let $\mathbb{Q}_\alpha$ be the poset from Case 2 of the same paper. Note that in this case $\mathbb{Q}_\alpha$ adds a dominating real. In either case A_α is defined as in [4]. With this the definition of $\mathbb{P}_{\omega_2}$ is complete. Following the proof of the original iteration, one can show that $\mathbb{P}_{\omega_2}$ adds a Δ^1_3-definable wellorder of the reals. Note that in $V^{\mathbb{P}_{\omega_2}}$ we have $\mathrm{add}(\mathcal{M}) = \aleph_2$, since cofinally often we add dominating and Cohen reals. To show that $\mathrm{cov}(\mathcal{N})$ remains small, i.e. that random reals are not added, use the fact that all iterands are σ-centered and [2, Theorems 6.5.30, 6.5.29]. We should point out that the coding techniques allow one to obtain the consistency of the existence of a Δ^1_3 wellorder of the reals with $\mathrm{add}(\mathcal{M}) = \aleph_3$ and $\mathrm{cov}(\mathcal{N}) = \aleph_1$. □

Theorem 4.19 *The constellation determined by* $\mathrm{cof}(\mathcal{M}) = \aleph_1$ *and* $\mathrm{non}(\mathcal{N}) = \aleph_2$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For each α let $\dot{\mathbb{Q}}^0_\alpha$ be the poset S_{g,g^*} defined in [2, Section 7.3.C]. Note that $V^{S_{g,g^*}} \models V \cap 2^\omega \in \mathcal{N}$. Thus clearly $V^{\mathbb{P}_{\omega_2}} \models \mathrm{non}(\mathcal{N}) = \aleph_2$. Now $\mathrm{cof}(\mathcal{N}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$. Thus it is sufficient to show that both $\mathfrak{d}$ and $\mathrm{non}(\mathcal{M})$ remain small in the final generic extension. However S_{g,g^*} is ${}^\omega\omega$-bounding and preserves $\sqsubseteq^{Cohen}$. Then theorems 2.6 and 2.11 imply that $\mathfrak{d} = \mathrm{non}(\mathcal{M}) = \aleph_1$ in $V^{\mathbb{P}_{\omega_2}}$. □

Theorem 4.20 *The constellation determined by* $\mathrm{non}(\mathcal{N}) = \mathfrak{b} = \aleph_2$ *and* $\mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}^0_\alpha$ be S_{g,g^*} and for α odd, let $\dot{\mathbb{Q}}^0_\alpha$ be the Laver forcing LT. Since all iterands are S-(g, g^*)-bounding, by theorem 2.17 $\mathbb{P}_{\omega_2}$ is S-(g, g^*)-bounding, which implies (see [2, Lemma 7.2.15]) that no real in $V^{\mathbb{P}_{\omega_2}}$ is Cohen or random over V. Therefore $\mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \aleph_1$ in $V^{\mathbb{P}_{\omega_2}}$. Recall also that $\Vdash_{S_{g,g^*}} \Vdash 2^\omega \cap V \in \mathcal{N}$ and LT adds a dominating real. □

Theorem 4.21 *The constellation determined by* $\mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \aleph_2$ *and* $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}^0_\alpha$ be $\mathrm{PT}_{f,g}$ and for α odd, let $\dot{\mathbb{Q}}^0_\alpha$ be S_{g,g^*}. Since $\Vdash_{\mathrm{PT}_{f,g}} 2^\omega \cap V \in \mathcal{M}$ and $\Vdash_{S_{g,g^*}} 2^\omega \cap V \in N$, we have $V^{\mathbb{P}_{\omega_2}} \models \mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \aleph_2$. All iterands are S-(f, h)-bounding and ${}^\omega\omega$-bounding, which implies that in $V^{\mathbb{P}_{\omega_2}}$ there are no random reals over V and the ground model reals form a dominating family. □

Theorem 4.22 *The constellation determined by* $\mathfrak{b} = \aleph_2$ *and* $\mathrm{non}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For every α let $\dot{\mathbb{Q}}^0_\alpha$ be the Laver forcing LT. Since LT adds a dominating function clearly $\mathfrak{b} = \aleph_2$. Since LT and Sacks coding S-preserve-$\sqsubseteq^{random}$, by Theorem 2.10 the ground model reals $V \cap 2^\omega$ are not null in $V^{\mathbb{P}_{\omega_2}}$. Since LT and Sacks coding have the Laver property, they are (f, g)-bounding which implies that the iteration does not add random reals. □

Theorem 4.23 *The constellation determined by* $\mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{N}) = \aleph_2$ *and* $\mathfrak{d} = \aleph_1$ *is consistent with the existence of a* Δ^1_3 *wellorder of the reals.*

Proof For α even let $\dot{\mathbb{Q}}^0_\alpha$ be the forcing notion S_{g,g^*} defined in [2, Section 7.3.C] and for α odd let $\dot{\mathbb{Q}}^0_\alpha$ be the random real forcing $\mathbb{B}$. Since S_{g,g^*} makes the ground model reals a null set, $V^{\mathbb{P}_{\omega_2}} \models \mathrm{non}(\mathcal{N}) = \aleph_2$. Clearly $\mathrm{cov}(\mathcal{N})$ is large in the final extension and since all iterands are ${}^\omega\omega$-bounding the ground model reals remain a witness to $\mathfrak{d} = \aleph_1$ in $V^{\mathbb{P}_{\omega_2}}$. □

5 Questions

We would like to conclude with some open questions. It is of interest whether all of the constellations can in fact be obtained without the existence of a Δ^1_3 wellorder of the reals. Note that this would follow if one could simultaneously have that all Δ^1_3 sets enjoy some regularity property that conflicts a Δ^1_3 wellorder. Can we even guarantee that there are no projective wellorders at all? Another direction is the question whether an assignment of larger values to the cardinal invariants in the Cichón diagram is consistent with the existence of a Δ^1_3 wellorder. What about constellations in which the invariants have more than two distinct values? Are those consistent with the existence of a Δ^1_3 wellorder of the reals?

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