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(will be inserted by the editor)

Co-analytic mad families and definable wellorders

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19 August 2012

Abstract We show that the existence of a Π^1_1-definable mad family is consistent with the existence of a Δ^1_3-definable well-order of the reals and $\mathfrak{b} = \mathfrak{c} = \aleph_3$.

Keywords Projective wellorders · projective mad families · large continuum

Mathematics Subject Classification (2000) 03E15 · 03E17 · 03E35

1 Introduction

An *almost disjoint* (*a.d.*) family $\mathcal{A}$ is a collection of infinite subsets of ω, such that $|a \cap b| < \omega$ for all $a, b \in \mathcal{A}$. A *maximal almost disjoint* (*mad*) family is an infinite a.d. family which is maximal with regard to this property, i.e., $\forall a \, \exists b \in \mathcal{A} \; (|a \cap b| = \omega)$. Mad families have been studied from a variety of perspectives: for example, the size of the least mad family that can possibly exist is the cardinal characteristic $\mathfrak{a}$, and its value has been shown to be independent of ZFC. Another perspective is the descriptive set-theoretic one, where one looks at the possible complexity of mad families (as subsets of $[\omega]^\omega$). This investigation has been carried out in a number of results, and we briefly summarize its history.

Theorem 1.1 (Mathias [15])**.** *There are no analytic mad families.*

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In the constructible universe L, it is easy to construct Σ^1_2-definable mad families. Arnold Miller proved a seemingly stronger result.

Theorem 1.2 (Miller [16]). *In L, there is a Π^1_1 mad family.*

The above result has recently been superseded by Asger Törnquist:

Theorem 1.3 (Törnquist [18]). *The following are equivalent:*

1. *There is a Σ^1_2 mad family.*
2. *There is a Π^1_1 mad family.*

Combining this theorem with well-known facts about constructing Σ^1_2-definable mad families in L which are preserved by iterations of some standard forcing notions (among which Cohen, random, Sacks and Miller forcing), one can easily see that the existence of a Π^1_1 mad family is consistent with $\neg$CH. On the other hand, the following was proved in [9] (where $\mathfrak{b}$ is the *bounding number*, i.e., the least size of an unbounded family, and an *ω-mad family* is a mad family satisfying a stronger maximality requirement—see e.g. [13] for a definition).

Theorem 1.4 (Friedman & Zdomskyy). *It is consistent that $\mathfrak{b} = \mathfrak{c} = \aleph_2$ and there exists a Π^1_2 ω-mad family.*

This was further extended in [7]:

Theorem 1.5 (Fischer, Friedman & Zdomskyy). *It is consistent that $\mathfrak{b} = \mathfrak{c} = \aleph_3$, there exists a Π^1_2 ω-mad family and a Δ^1_3-definable well-order of the reals.*

Methods for obtaining models with large continuum together with a Δ^1_3-definable wellorder have been developed by Jensen and Solovay [12], by Harrington [10] and by Friedman [8]; it is an ongoing project to determine to what extent the Δ^1_3 wellorder is compatible with certain other properties of the model (such as a cardinal inequality or the existence of other projective objects, cf. [5,7,6]).

Dropping the "ω"-requirement, Theorem 1.4 was improved in [4]:

Theorem 1.6 (Brendle & Khomskii). *For any regular uncountable cardinal κ, it is consistent that $\mathfrak{b} = \mathfrak{c} = \kappa$ and there exists a Π^1_1 mad family.*

The present paper is concerned with the following question: to what extent can Theorem 1.5 be merged with Theorem 1.6? Note that we have no chance of obtaining a Π^1_1 ω-mad family together with $\mathfrak{b} > \aleph_1$ (the reason is that an ω-family does not contain a perfect set by [17], so a Π^1_1 ω-mad family must be completely contained in L), so the "ω"-requirement must certainly be dropped. Taking that into account, we do indeed succeed in proving an optimal result extending both Theorem 1.5 and Theorem 1.6.

Theorem 1.7 (Main Theorem). *It is consistent that $\mathfrak{b} = \mathfrak{c} = \aleph_3$, there exists a Π^1_1 mad family, and a Δ^1_3-definable well-order of the reals.*

For the proof of this theorem, we use a combination of the techniques for constructing Δ^1_3 well-orders, as presented in [7], and the techniques from [4] for constructing a Π^1_1 mad family in models where $\mathfrak{b}$ is large. Most of the work involves overcoming two main obstacles:

1. showing that a version of *almost disjoint coding* has a nice preservation property, and
2. dealing with iterations longer than length $\aleph_1$.

The first obstacle will be solved in Lemma 2.4 and the second one will be solved by making use of the $\diamondsuit$-principle, in Lemma 4.1.

Note that, while our main theorem is formulated as an optimal generalization of Theorems 1.5 and 1.6, it would be easy to modify the proof so that it yields the same result with $\mathfrak{b} = \mathfrak{c} = \aleph_2$ instead of $\aleph_3$. The only difference would be a straightforward simplification of the coding mechanism. We also conjecture that the same result holds for $\mathfrak{b} = \mathfrak{c} = \kappa$ for any uncountable regular κ, although that would require a substantial change to the coding mechanism, and it is still not completely clear whether that can be done.

This paper is structured as follows: in Section 2, we give the preliminary definitions, review the main methods of [4] and [7], and introduce a different version of "almost disjoint coding". In Section 3, we review the preparatory forcing construction from [7] and prove that the $\diamondsuit$-principle is preserved after the preparation. Finally, in Section 4, we combine these efforts and obtain a proof of the Main Theorem.

2 Preliminaries

We start by summarizing the main tools behind the result of [4]. One of the central concepts there was considering mad families constructed from perfect a.d. families, and *preserving the maximality of the re-interpreted family* by forcing, as opposed to the more classical concept of preserving a mad family *directly.*

Definition 2.1. *A set $\mathcal{A} \subseteq [\omega]^\omega$ is called an* $\aleph_1$-perfect mad family *if $\mathcal{A} = \bigcup_{\alpha<\aleph_1} A_\alpha$ where each A_α is a perfect a.d. set and $\mathcal{A}$ is a mad family. For a forcing $\mathbb{P}$, such a family $\mathcal{A}$ is said to be* $\mathbb{P}$-indestructible *if in the generic extension $V[G]$ by $\mathbb{P}$, $A^{V[G]} := \bigcup_{\alpha<\aleph_1} A_\alpha^{V[G]}$ is a mad family.*

The method of [4] involved the construction of an $\aleph_1$-perfect mad family in L, which had a Σ^1_2 definition and moreover was indestructible (in the sense of Definition 2.1) by the κ-iteration of Hechler forcing with finite support, for κ being any uncountable regular cardinal. We now briefly review that construction.

For $\alpha < \aleph_1$, let $P^\alpha := \{P^\alpha_\sigma \mid \sigma \in \omega^{<\omega}\}$ be an infinite partition of some (unspecified) domain $D_\alpha \in [\omega]^\omega$ into infinite sets, indexed by finite sequences σ. For each $\sigma \in \omega^{<\omega}$, let $\{p^\alpha_\sigma(0), p^\alpha_\sigma(1), p^\alpha_\sigma(2), \dots\}$ be the increasing enumeration of P^α_σ. For each $f \in {}^\omega\omega$, let $\Phi^\alpha(f) := \{p^\alpha_{f\restriction n}(f(n)) \mid n \in \omega\}$ and let $A_\alpha := \{\Phi^\alpha(f) \mid f \in {}^\omega\omega\}$. Then A_α is an almost disjoint subfamily of $[D_\alpha]^\omega$ of size $2^{\aleph_0}$. Furthermore A_α is a perfect set in the natural topology of $[D_\alpha]^\omega$ (since Φ is a homeomorphism between ω^ω and A_α).

The idea is then to construct, by induction on $\alpha < \aleph_1$, a sequence of such partitions P^α, each of them giving rise to a perfect a.d. set A_α, and to make sure that the union $\mathcal{A} := \bigcup_{\alpha<\aleph_1} A_\alpha$ becomes a mad family. If the construction takes place in L, it is easy to make it Σ^1_2-definable. To guarantee preservation by Hechler forcing, the following essential property was used:

Definition 2.2. *A forcing* $\mathbb{P}$ strongly preserves splitting reals *(abbreviated by "s.p.s."), if for every* $\mathbb{P}$*-name* $\dot{a}$ *for an element of* $[\omega]^\omega$*, there is a sequence* $\{a_n : n \in \omega\}$ *of elements of* $[\omega]^\omega$*, such that if* $z \in [\omega]^\omega$ *splits all* a_n*'s, then* $\Vdash_{\mathbb{P}}$ *"*$\check{z}$ *splits* $\dot{a}$*".*

The Hechler partial order satisfies the s.p.s.-property by [1]. Moreover, the s.p.s.-property is preserved by iterations of ccc forcings with finite support (see [2, Proposition 3.10]).

We now state the Main Lemma from [4], involved in the induction step of the construction (in the original Lemma, an ideal on ω was also generated for technical reasons, but we leave it out here since it would only make the presentation more difficult). The notations P^β, A_β, Φ^β etc. refer to the objects described above.

Lemma 2.3 (Main Lemma, [4])**.** *Let* M *be a countable model of set theory such that* $P^\beta \in M$ *for all* $\beta < \alpha$*. Assume that for all* $\beta \neq \beta' < \alpha$ *and for all* $f, g \in {}^\omega\omega$*, the set* $\Phi^\beta(f) \cap \Phi^{\beta'}(g)$ *is finite (i.e.,* $\bigcup_{\beta<\alpha} A_\beta$ *is an a.d. family).*

Then there exists a new partition P^α *(of some domain* D_α*), lying outside* M*, which satisfies the following properties:*

1. *For every* $f, h \in {}^\omega\omega$ *and every* $\beta < \alpha$*,* $\Phi^\beta(f) \cap \Phi^\alpha(h)$ *is finite (i.e.,* $\bigcup_{\beta\leq\alpha} A_\beta$ *is still a.d.)*
2. *For every* $Y \in M$*, if* Y *is almost disjoint from* $\Phi^\beta(f)$ *for all* $f \in {}^\omega\omega$ *and all* $\beta < \alpha$*, then there exists an* $h \in {}^\omega\omega$ *such that* $\Phi^\alpha(h) \subseteq Y$*.*
3. *Suppose* $V' \supseteq V$ *is a model of set theory,* $M' \supseteq M$ *is a countable model with* $M' \in V'$*, and every real in* V *which is splitting over* M *is still splitting over* M'*. Then for every* $Y \in M'$*, if* Y *is almost disjoint from* $\Phi^\beta(f)$ *for every* $f \in {}^\omega\omega$ *in* V' *and every* $\beta < \alpha$*, then there exists an* $h \in {}^\omega\omega$ *in* V' *such that* $V' \vDash \Phi^\alpha(h) \subseteq Y$ *(i.e., condition 2 holds relativized to* V' *and* M'*.)*

Clearly, the above lemma can be applied with $V' = V[G]$, and $M' = M[G]$ being generic extensions via some forcing that satisfies the s.p.s. property. This Lemma will be the crucial tool in our inductive construction of the mad family in Section 4.

Next, we shift our attention to the Δ^1_3-definable well-order of the reals. As a Σ^1_2-definable well-order implies that every real is constructible (see e.g. [11, Theorem 25.39]), a Δ^1_3 well-order is optimal in the presence of $\neg$CH. A (boldface) $\boldsymbol{\Delta}^1_3$ well-order together with $\neg$CH was first obtained by Harrington [10], and Sy Friedman improved this result by establishing the consistency of a (lightface) Δ^1_3 well-order of the reals together with $\mathfrak{c} = \aleph_2$. Different methods of obtaining large continuum, Δ^1_3 well-orders, and the existence of certain combinatorial objects on the reals have recently been developed in [5, 7, 6] (dealing with cardinal inequalities, ω-mad families, and MA, respectively).

In this paper we will mostly be using the methods from [7]. The final model will be obtained as a two-step forcing extension of L. In the first stage (the "preliminary stage") the universe is prepared in a special way, by adding certain subsets of $\aleph_1$ and $\aleph_2$ but no new reals. We will denote this intermediate extension by L^*, and the forcing leading up to it by $\mathbb{P}^*$. Note that since no new reals have been added, L^* still satisfies many properties of L, such as having a Σ^1_2-good well-ordering of the reals.

In the next stage (the "coding stage"), new reals are added to L^*, by a finite support iteration of length $\aleph_3$, consisting of σ-centered forcing posets. This iteration simultaneously makes sure that $\mathfrak{b} = \aleph_3$ and that a Δ^1_3 well-ordering of the reals exists. In [7], the Π^1_2-definable mad family was explicitly added by this forcing as well. In our situation, we would like to preserve an $\aleph_1$-perfect mad family defined in L^* instead. We would like to simulate the proof in [4], but for that we need two ingredients: the s.p.s.-property of the forcing, and a way to deal with iterations of length longer than $\aleph_1$. The next theorem deals with the first ingredient.

Recall that one of the central methods in the "coding stage" is *almost disjoint coding*, a technique which allows subsets of ω_1 to be coded by reals in a generic extension. We show that this can be done by a forcing having the s.p.s.-property.

Let $\vec{C} := \{c_\alpha : \alpha < \aleph_1\}$ be a fixed, definable (e.g. closed) family of a.d. sets, and let $A \subseteq \omega_1$ be an arbitrary set. Let $\mathcal{I}_A$ be the ideal on ω generated by the a.d. family $\{c_\alpha : \alpha \in A\}$, let $\mathcal{I}^+_A$ denote $\mathcal{I}_A$-positive sets and $\mathcal{F}_A$ the corresponding filter. While the standard almost disjoint coding can be seen as a Mathias partial order with the filter $\mathcal{F}_A$, we will use a Laver-like partial order instead. Precisely, we prove the following:

Lemma 2.4. *For any given $\vec{C}$ and $A \subseteq \omega_1$, there exists a σ-centered forcing, which we shall denote by $\mathbb{L}_A(\vec{C})$, such that*

1. *$\mathbb{L}_A(\vec{C})$ adds a dominating real,*
2. *$\mathbb{L}_A(\vec{C})$ satisfies the s.p.s.-property, and*
3. *$\mathbb{L}_A(\vec{C})$ adds a generic real $\dot{x}_G$ with the following property:*
   (a) if $\alpha \in A$ then $\Vdash |\mathrm{ran}(\dot{x}_G) \cap c_\alpha| < \omega$, and
   (b) if $\alpha \notin A$ then $\Vdash |\mathrm{ran}(\dot{x}_G) \cap c_\alpha| = \omega$.
   Consequently, $\Vdash$ "$\dot{x}_G$ encodes A".

Proof. Let $\mathbb{L}_A(\vec{C})$ be the Laver partial order with filter $\mathcal{F}_A$, i.e., the partial order consisting of all trees T such that for any $t \in T$ longer than $\mathrm{stem}(T)$, we have $\mathrm{Succ}_T(t) := \{n \mid t^\frown\langle n\rangle \in T\} \in \mathcal{F}_A$; the ordering is inclusion.

It is clear that this forcing is σ-centered, and to see that it adds a dominating real, simply note that for any $t \in T$, if $\mathrm{Succ}_T(t) \in \mathcal{F}_A$ then also $\mathrm{Succ}_T(t) \setminus m \in \mathcal{F}_A$ for any finite m. To verify that it has the s.p.s.-property, we use a result of Brendle and Hrušák [3]. We need some definitions:

– An ideal $\mathcal{I}$ on ω is *countably tall* if for any sequence $\{a_n \mid n < \omega\}$ of infinite subsets of ω, there is $b \in \mathcal{I}$ such that $|a_n \cap b| = \omega$ for every n.
– For two ideals $\mathcal{I}, \mathcal{J}$, write $\mathcal{J} \leq_K \mathcal{I}$ (*$\mathcal{J}$ is Katetov-reducible to $\mathcal{I}$*) iff there is an $f : \omega \to \omega$ s.t. $\forall a\ (a \in \mathcal{J} \to f^{-1}[a] \in \mathcal{I})$.

A recent result from [3, Proposition 1] then states the following: Let $\mathcal{I}$ be an arbitrary ideal and $\mathcal{F}$ the corresponding filter. Then the following are equivalent:

1. For all $X \in \mathcal{I}^+$ and every $\mathcal{J} \leq_K \mathcal{I}{\restriction}X$, $\mathcal{J}$ is not countably tall.
2. Laver forcing with the filter $\mathcal{F}$ has the s.p.s.-property.

The argument for the proof of this result is quite similar to the one for Hechler forcing—indeed Hechler forcing can be seen as Laver with the cofinite filter. To prove that $\mathbb{L}_A(\vec{C})$ satisfies the s.p.s., it suffices to show that $\mathcal{I}_A$ satisfies clause 1 from above. The main point is that $\mathcal{I}_A$ itself is not countably tall.

Let $X \in \mathcal{I}_A^+$, let $\mathcal{J} \leq_K \mathcal{I}_A{\restriction}X$ and let $f : X \to \omega$ be the function witnessing this Katetov reduction. We have two cases:

Case 1: The set $\{\alpha \in A \mid |f``(c_\alpha \cap X)| = \omega\}$ is finite. Let $\{\alpha_1, \dots, \alpha_n\}$ enumerate it. Since $X \notin \mathcal{I}_A$, the remaining set $Y := X \setminus (c_{\alpha_1} \cup \dots \cup c_{\alpha_n})$ is also not in $\mathcal{I}_A$. Let $Y' := f``Y$. Then $Y' \notin \mathcal{J}$, so, in particular, Y' is infinite. We claim that no infinite subset $Z \subseteq Y'$ can be in $\mathcal{J}$, which will witness the fact that $\mathcal{J}$ is not countably tall (in fact it will not even be tall).

Towards a contradiction, let $Z \subseteq Y'$ be an infinite set in $\mathcal{J}$. Then $f^{-1}[Z] \in \mathcal{I}_A$, and $f^{-1}[Z] \cap Y$ is an infinite set, also in $\mathcal{I}_A$. So $f^{-1}[Z] \cap Y$ must be almost covered by some finitely many $c_{\beta_1}, \dots, c_{\beta_k}$, none of which can be among the c_{α_i}'s. Therefore, Z is almost covered by finitely many sets of the form $f``(c_{\beta_j} \cap Y)$, where $c_{\beta_j} \neq c_{\alpha_i}$ for any i. But by assumption, all such sets were finite, contradicting that Z is infinite.

Case 2: The set $\{\alpha \in A \mid |f``(c_\alpha \cap X)| = \omega\}$ is infinite. Pick a countable sequence $\{\alpha_n \mid n < \omega\}$ from it, and let $a_n := f``(c_{\alpha_n} \cap X)$. We claim that $\{a_n \mid n < \omega\}$ is a witness to the fact that $\mathcal{J}$ is not countably tall. Let $b \in \mathcal{J}$ be arbitrary. By assumption, $b' := f^{-1}[b] \in \mathcal{I}_A$. This means that there are $\beta_1, \dots \beta_k \in A$ such that $b' \subseteq^* c_{\beta_1} \cup \dots \cup c_{\beta_k}$. But then b' cannot have infinite intersection with infinitely many of the c_{α_n}'s, since otherwise some c_{β_i} and some c_{α_n}, with $\beta_i \neq \alpha_n$, would have infinite intersection, contradicting their mutual almost disjointness. Therefore, for some n, b' has only finite intesection with c_{α_n}. But then $b = f``(b')$ has finite intersection with a_n, proving that $\mathcal{J}$ is not countably tall.

It remains to show that $\mathbb{L}_A(\vec{C})$ can be used for a.d. coding purposes, i.e., condition (3) from the theorem.

(a) Let $\alpha \in A$ and $T \in \mathbb{L}_A(\vec{C})$. Inductively let $S \leq T$ be obtained by pruning the tree and removing c_α from every splitting node, i.e., making sure that $\mathrm{Succ}_S(t) := \mathrm{Succ}_T(t) \setminus c_\alpha$ for every $t \in S$. Since $\omega \setminus \mathrm{Succ}_T(t)$

is in $\mathcal{I}_A$ and $\alpha \in A$, $\omega \setminus \mathrm{Succ}_S(t)$ is also in $\mathcal{I}_A$, so the tree S is a valid $\mathbb{L}_A(\vec{C})$-condition. Moreover, for all n above the stem, $S \Vdash n \notin c_\alpha$. Hence $S \Vdash |\mathrm{ran}(\dot{x}_G) \cap c_\alpha| < \omega$.

(b) Let $\alpha \notin A$, $T \in \mathbb{L}_A(\vec{C})$ and $n \in \omega$ be given. Let $t := \mathrm{stem}(T)$ and consider $\mathrm{Succ}_T(t) \in \mathcal{F}_A$. Since $\alpha \notin A$ and the collection $\{c_\beta \mid \beta < \aleph_1\}$ was a.d., clearly $c_\alpha \notin \mathcal{I}_A$. But then $\mathrm{Succ}_T(t) \cap c_\alpha$ is infinite and so we may pick $m \geq n$ from this set. Then letting $S \leq T$ be such that $\mathrm{stem}(S) = t^\frown\langle m\rangle$ we have $S \Vdash m \in \mathrm{ran}(\dot{x}_G)$.

As a result, $\dot{x}_G$ codes A as we wanted. □

3 The preliminary stage, $\diamondsuit$ and $\diamondsuit'$

In this section we review the preliminary forcing construction leading from L to L^*, and verify that the $\diamondsuit$-principle is valid in L^*. Most of the exposition here follows closely that of [7], although many details are left out. We start by defining the preliminary forcing $\mathbb{P}^* = \mathbb{P}^0 * \dot{\mathbb{P}}^1 * \dot{\mathbb{P}}^2$.

A transitive ZF$^-$ model $\mathcal{M}$ is *suitable* if $\omega_3^{\mathcal{M}}$ exists and $\omega_3^{\mathcal{M}} = \omega_3^{L^{\mathcal{M}}}$. If $\mathcal{M}$ is suitable then also $\omega_1^{\mathcal{M}} = \omega_1^{L^{\mathcal{M}}}$ and $\omega_2^{\mathcal{M}} = \omega_2^{L^{\mathcal{M}}}$.

Fix a $\diamondsuit_{\omega_2}(cof(\omega_1))$ sequence $\langle G_\xi \mid \xi \in \omega_2 \cap cof(\omega_1)\rangle$ which is Σ_1-definable over L_{ω_2}. For $\alpha < \omega_3$, let W_α be the $<_L$-least subset of ω_2 coding α, and for $1 < \alpha < \omega_3$ let $S_\alpha = \{\xi \in \omega_2 \cap cof(\omega_1) \mid G_\xi := W_\alpha \cap \xi \neq \emptyset\}$. Then $\vec{S} = \langle S_\alpha \mid 1 < \alpha < \omega_3\rangle$ is a sequence of stationary subsets of $\omega_2 \cap cof(\omega_1)$, which are mutually almost disjoint. Let $S_{-1} := \{\xi \in \omega_2 \cap \mathrm{cof}(\omega_1) \mid G_\xi = \emptyset\}$. Note that S_{-1} is a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$ which is disjoint from all S_α's.

Step 0. For every α such that $\omega_2 \leq \alpha < \omega_3$ "shoot a club" C_α disjoint from S_α via the poset $\mathbb{P}^0_\alpha$, consisting of all closed subsets of ω_2 which are disjoint from S_α ordered by end-extension, and let $\mathbb{P}^0 = \prod_{\alpha<\omega_3} \mathbb{P}^0_\alpha$ be the direct product of the $\mathbb{P}^0_\alpha$'s with supports of size ω_1, where for $\alpha \in \omega_2$, $\mathbb{P}^0_\alpha$ is the trivial poset. Then $\mathbb{P}^0$ is countably closed, ω_2-distributive (the proof of which uses the stationarity of S_{-1}) and ω_3-c.c.

Step 1. We begin by fixing some notation. Whenever $k \in \omega$, X is a set of ordinals and $j \in k$, let $I^k_j(X) = \{\gamma \mid k \cdot \gamma + j \in X\}$. In particular, let $Even(X) = I^2_0(X) = \{\gamma \mid 2 \cdot \gamma \in X\}$. For every $\alpha < \omega_3$ let D_α be a subset of ω_2 which codes the triple $\langle C_\alpha, W_\alpha, W_\gamma\rangle$ where γ is the largest limit ordinal $\leq \alpha$, precisely: $I^3_0(D_\alpha) = C_\alpha$, $I^3_1(D_\alpha) = W_\alpha$ and $I^3_2(D_\alpha) = W_\gamma$. Let

$$E_\alpha = \{\mathcal{M} \cap \omega_2 \mid \mathcal{M} \prec L_{\alpha+\omega_2+1}[D_\alpha],\ \omega_1 \cup \{D_\alpha\} \subseteq \mathcal{M}\}.$$

Then E_α is a club on ω_2. Choose $Z_\alpha \subseteq \omega_2$ such that $Even(Z_\alpha) = D_\alpha$ and if $\beta < \omega_2$ is $\omega_2^{\mathcal{M}}$ for some suitable model $\mathcal{M}$ such that $Z_\alpha \cap \beta \in \mathcal{M}$, then $\beta \in E_\alpha$. Then we have:

$(*)_\alpha$: If $\beta < \omega_2$, $\mathcal{M}$ is a suitable model such that $\omega_1 \subset \mathcal{M}$, $\omega_2^{\mathcal{M}} = \beta$, and $Z_\alpha \cap \beta \in \mathcal{M}$, then $\mathcal{M} \vDash \psi(\omega_2, Z_\alpha \cap \beta)$, where $\psi(\omega_2, X)$ is the formula "$Even(X)$ codes a triple $(\bar{C}, \bar{W}, \bar{\bar{W}})$, where $\bar{W}$ and $\bar{\bar{W}}$ are the $<_L$-least codes of ordinals $\bar{\alpha}, \bar{\bar{\alpha}} < \omega_3$ such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$ and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\alpha}}$".

Similarly to $\vec{S}$, define a sequence $\vec{A} = \langle A_\xi \mid \xi < \omega_2 \rangle$ of stationary subsets of ω_1 which are mutually almost disjoint, using the "standard" $\diamondsuit$-sequence. Code Z_α by a subset X_α of ω_1 with the poset $\mathbb{P}^1_\alpha$ consisting of all pairs $\langle s_0, s_1 \rangle \in [\omega_1]^{<\omega_1} \times [Z_\alpha]^{<\omega_1}$ where $\langle t_0, t_1 \rangle \leq \langle s_0, s_1 \rangle$ iff s_0 is an initial segment of t_0, $s_1 \subseteq t_1$ and $t_0 \backslash s_0 \cap A_\xi = \emptyset$ for all $\xi \in s_1$ (note that this is closely related to the a.d. coding discussed in Section 2, but deals with coding subsets of ω_2 by subsets of ω_1). Then X_α satisfies the following condition:

$(**)_\alpha$: If $\mathcal{M}$ is a suitable model such that $\{X_\alpha\} \cup \omega_1 \subset \mathcal{M}$, then $\mathcal{M} \vDash \phi(\omega_1, \omega_2, X_\alpha)$, where $\phi(\omega_1, \omega_2, X)$ is the formula: "Using the sequence $\vec{A}$, X almost disjointly codes a subset $\bar{Z}$ of ω_2, such that $Even(\bar{Z})$ codes a triple $(\bar{C}, \bar{W}, \bar{\bar{W}})$, where $\bar{W}$ and $\bar{\bar{W}}$ are the $<_L$-least codes of ordinals $\bar{\alpha}, \bar{\bar{\alpha}} < \omega_3$ such that $\bar{\bar{\alpha}}$ is the largest limit ordinal not exceeding $\bar{\alpha}$ and $\bar{C}$ is a club in ω_2 disjoint from $S_{\bar{\alpha}}$".

Let $\mathbb{P}^1 = \prod_{\alpha<\omega_3} \mathbb{P}^1_\alpha$, where $\mathbb{P}^1_\alpha$ is the trivial poset for all $\alpha \in \omega_2$, with countable support. Then $\mathbb{P}^1$ is countably closed and has the ω_2-c.c.

Step 2. Finally we force a "localization" of the X_α's. Fix ϕ as in $(**)_\alpha$ and define the poset $\mathcal{L}_k(X, X')$ as in [7, Definition 1]. That is, let $X, X' \subset \omega_1$ be such that $\phi(\omega_1, \omega_2, X)$ and $\phi(\omega_1, \omega_2, X')$ hold in any suitable model $\mathcal{M}$ with $\omega_1^{\mathcal{M}} = \omega_1^L$ containing X and X', respectively. Then let $\mathcal{L}(X, X')$ be the poset of all functions $r : |r| \to 2$, where the domain $|r|$ of r is a countable limit ordinal such that:

1. if $\gamma < |r|$ then $\gamma \in X$ iff $r(3\gamma) = 1$
2. if $\gamma < |r|$ then $\gamma \in X'$ iff $r(3\gamma + 1) = 1$
3. if $\gamma \leq |r|$, $\mathcal{M}$ is a countable suitable model containing $r\restriction\gamma$ as an element and $\gamma = \omega_1^{\mathcal{M}}$, then $\mathcal{M} \vDash \phi(\omega_1, \omega_2, X \cap \gamma) \wedge \phi(\omega_1, \omega_2, X' \cap \gamma)$.

The ordering is end-extension.

For every $\alpha \in Lim(\omega_3)$ and $m \in \omega$, let $\mathbb{P}^2_{\alpha+m} = \mathcal{L}(X_{\alpha+m}, X_\alpha)$. Let

$$\mathbb{P}^2 = \prod_{\alpha \in Lim(\omega_3)} \prod_{m \in \omega} \mathbb{P}^2_{\alpha+m}$$

with countable supports. In $L^{\mathbb{P}^0 * \mathbb{P}^1}$, the poset $\mathbb{P}^2$ has the ω_2-c.c. Also note that $\mathbb{P}^2_{\alpha+m}$ produces a generic function in the space 2^{ω_1} (of $L^{\mathbb{P}^0 * \mathbb{P}^1}$), which is the characteristic function of a subset $Y_{\alpha+m}$ of ω_1 with the following property:

$(***)_\alpha$: For every $\beta < \omega_1$ and any suitable $\mathcal{M}$ such that $\omega_1^{\mathcal{M}} = \beta$ and $Y_{\alpha+m} \cap \beta$ belongs to $\mathcal{M}$, we have $\mathcal{M} \models \phi(\omega_1, \omega_2, X_{\alpha+m} \cap \beta) \wedge \phi(\omega_1, \omega_2, X_\alpha \cap \beta)$.

Now we let $\mathbb{P}^* := \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ be the result of combining these three generic extensions, and use L^* to denote the intermediary extension $L^{\mathbb{P}^*}$.

Next, we want to show that $\diamondsuit$ holds in L^*. Since we have added new subsets of ω_1, this is not a priori obvious. To prove that this is the case, we use the related $\diamondsuit'$-principle, a version of $\diamondsuit$ due to Kunen, in which we allow countably many possibilities at stage α to capture sets.

Definition 3.1. *A sequence* $\{S_\alpha \mid \alpha < \omega_1\}$ *is a* $\diamondsuit'$-sequence *if* $S_\alpha = \{S_\alpha^n \mid n < \omega\}$ *such that* $S_\alpha^n \subseteq \alpha$ *for all* n*, and if for all* $S \subseteq \omega_1$*, the set* $\{\alpha \mid \exists n\, (S \cap \alpha = S_\alpha^n)\}$ *is stationary.*

Lemma 3.2. *In* L^**, there exists a* $\diamondsuit'$*-sequence which is* Σ_1*-definable over* L_{ω_1}.

Proof. We define the sequence in L, and show that it is preserved by $\mathbb{P}^*$. For $\alpha < \omega_1 (= \omega_1^L)$, let $\beta(\alpha)$ be the least ordinal β such that $L_\beta \models \mathrm{ZF}^- +$ (α is countable). Let $D'_\alpha := \{A \subseteq \alpha \mid A \in L_{\beta(\alpha)}\}$. We claim that $\{D'_\alpha \mid \alpha < \omega_1\}$ is a $\diamondsuit'$-sequence even after forcing with $\mathbb{P}^*$.

So, let $p_0 \in \mathbb{P}^*$, let $\dot{X}$ be a $\mathbb{P}^*$-name for a subset of ω_1 and $\dot{C}$ a $\mathbb{P}^*$-name for a closed unbounded subset of ω_1. Let $\mathcal{N}$ be the least countable elementary submodel of some large L_Θ such that $p_0, \dot{X}, \dot{C}$ are elements of $\mathcal{N}$. Let $\bar{\mathcal{N}}$ be the transitive collapse of $\mathcal{N}$.

As in the proof of [7, Lemma 1], get an extension p_1 of p_0 which meets all dense sets in $\mathcal{N}$ by considering a generic filter g over $\mathcal{N}$. Now let $\bar{g}$ be the image of g under the transitive collapse that maps $\mathcal{N}$ to $\bar{\mathcal{N}}$.

Then $\bar{g}$ is definable from an ω-enumeration of $\bar{\mathcal{N}}$ and (as $\mathcal{N}$ is the least countable elementary submodel of some L_Θ containing a certain finite set of parameters) there is such an ω-enumeration in $L_{\beta(\alpha)}$, where $\alpha = \mathcal{N} \cap \omega_1$. So $p_1 \Vdash \dot{X} \cap \alpha \in L_{\beta(\alpha)}$ and $p_1 \Vdash \alpha \in \dot{C}$. But then p_1 forces that the intersection $\{\alpha \mid \dot{X} \cap \alpha \in D'_\alpha\} \cap \dot{C}$ is non-empty, which completes the proof. □

To conclude, note that by [14, Theorem II 7.14], every $\diamondsuit'$-sequence gives rise to a $\diamondsuit$-sequence in a natural way. Consequently, there is a $\diamondsuit$-sequence in L^* which is Σ_1 definable over L_{ω_1}.

4 Constructing an s.p.s.-indestructible mad family in L^*.

We are now ready to prove the main theorem, using an inductive construction and Lemma 2.3. In [4], the method was to define a sequence $\{M_\alpha \mid \alpha <$

$\aleph_1\}$ of countable models covering all $\mathbb{D}_{\aleph_1}$-names for reals (where $\mathbb{D}_{\aleph_1}$ stands for the $\aleph_1$-iteration of Hechler forcing), while simultaneously constructing the perfect a.d. families A_α using Lemma 2.3. Condition 3 of the Lemma then guaranteed that the family $\mathcal{A} := \bigcup_{\alpha<\aleph_1} A_\alpha$ thus constructed was not destroyed by $\mathbb{D}_{\aleph_1}$, and an additional argument (involving the fact that Hechler forcing is Suslin ccc) then showed that the same must hold for $\mathbb{D}_\kappa$, where κ is any regular uncountable cardinal. Since we will need to deal with more complicated iterations, which are ccc but not Suslin, we need a different method for dealing with longer iterations, and we use the $\diamondsuit$-sequence for this purpose.

Lemma 4.1. *Let $\mathbb{P}$ be any ccc forcing notion satisfying the s.p.s.-property. Then there exists a $\mathbb{P}$-indestructible, $\aleph_1$-perfect, Σ^1_2-definable mad family in L^*. Moreover, in $(L^*)^{\mathbb{P}}$ this family still has a Σ^1_2 definition.*

Proof. First of all, note that we may assume, without loss of generality, that in L^* there exists a definable 5-dimensional version of $\diamondsuit$, namely, a sequence

$$\{(X_\alpha, E_\alpha, <_\alpha) \mid \alpha < \aleph_1\}$$

such that $X_\alpha \subseteq \omega_1$, $E_\alpha, <_\alpha \ \subseteq (\omega_1 \times \omega_1)$, and for every triple $(X, E, <)$, the set

$$\{\alpha \mid X \cap \alpha = X_\alpha, E \cap (\alpha \times \alpha) = E_\alpha \text{ and } (< \cap(\alpha \times \alpha)) = <_\alpha\}$$

is stationary. Fix such a sequence for the rest of the proof.

Definition 4.2. *We say that a triple $(X, E, <)$ "codes a* ZF$^-$ *model" iff*

1. *E and $<$ are binary relations on X,*
2. *(X, E) is well-founded and extensional,*
3. *$<$ well-orders X, and*
4. *$(X, E) \models$ ZF$^-$.*

We proceed by defining the $\aleph_1$-mad family, by induction on $\alpha < \aleph_1$, using the ideas described in Section 2. At each step, Lemma 2.3 is applied to produce the next partition P^α (of some domain D_α), giving rise to a perfect a.d. set A_α. Simultaneously, a sequence of countable transitive ZF$^-$ models $\{M_\alpha \mid \alpha < \aleph_1\}$ will be defined (note that the transitivity of the models is crucial in the current argument). Inductively, the following conditions will be guaranteed for all α:

1. $\langle M_\beta \mid \beta < \alpha\rangle \in M_\alpha$,
2. $\langle P^\beta \mid \beta < \alpha\rangle \in M_\alpha$,
3. $\bigcup_{\beta<\alpha} A_\beta$ is a.d.

We proceed with the inductive construction. At stage α, assume M_β and P^β have been defined, and the three inductive conditions are satisfied. To define M_α, consider two cases:

- **Case 1.** If $(X_\alpha, E_\alpha, <_\alpha)$ codes a countable ZF$^-$ model (in the sense of Definition 4.2), let M'_α be its transitive collapse. If, additionally, it so happens that $\langle M_\beta \mid \beta < \alpha\rangle \in M'_\alpha$ and $\left\langle P^\beta \mid \beta < \alpha\right\rangle \in M'_\alpha$, let $M_\alpha := M'_\alpha$.

– **Case 2.** If the above fails, then simply let M_α be the countable, transitive ZF^- model with $<_L$-least code, such that $\langle M_\beta \mid \beta < \alpha\rangle \in M_\alpha$ and $\langle P^\beta \mid \beta < \alpha\rangle \in M_\alpha$.

After that, we are in the right situation to apply Lemma 2.3 to the model M_α and the collection of partitions $\langle P^\beta \mid \beta < \alpha\rangle$, so we use it to construct a new partition P^α, picking the $<_L$-least one satisfying all the conditions.

This completes the inductive definition. We claim that $\mathcal{A} := \bigcup_{\alpha<\aleph_1} A_\alpha$ thus constructed is a $\mathbb{P}$-indestructible mad family. By the third inductive condition, it follows immediately that $\mathcal{A}$ is a.d., so let's focus on its maximality. Let G be $\mathbb{P}$-generic, and let Y be a new real in $L^*[G]$. Since $\mathbb{P}$ may have added many reals, by basic cardinality arguments we clearly cannot assume that Y is contained in some $M_\alpha[G]$. However, here we will use $\diamondsuit$ to get around this difficulty.

Claim 4.3. *For some $\alpha < \omega_1$, Y belongs to a generic extension of M_α via some forcing which has the s.p.s.-property.*

Proof. Let $\dot{Y}$ be a $\mathbb{P}$-name for Y. Let N be a countably closed, elementary submodel of some sufficiently large $\mathcal{H}_\theta$, with $|N| = \aleph_1$, containing $\mathbb{P}$, $\dot{Y}$, the entire sequences $\langle M_\alpha \mid \alpha < \omega_1\rangle$ and $\langle P^\alpha \mid \alpha < \omega_1\rangle$, and all the countable ordinals. Let E and $<$ be binary relations on ω_1 so that $(N, \in, <_{\mathcal{H}_\theta}) \cong (\omega_1, E, <)$ (here $<_{\mathcal{H}_\theta}$ refers to some natural well-order of N inherited from $\mathcal{H}_\theta$). Also, let $\langle N_\alpha \mid \alpha < \omega_1\rangle$ be a continuous sequence of countable elementary submodels of $\mathcal{H}_\theta$, converging to N. Note that if G is $\mathbb{P}$-generic, then, since $\mathbb{P}$ is ccc, $N[G]$ is a generic extension of N via $\mathbb{P}\cap N$ and $N_\alpha[G]$ is a generic extension of N_α via $\mathbb{P}\cap N_\alpha$.

Moreover, by continuity of the sequence $\langle N_\alpha \mid \alpha < \omega_1\rangle$, there are club-many α so that

$$(N_\alpha, \in, <_{\mathcal{H}_\theta}) \cong (\alpha,\ E\cap(\alpha\times\alpha), <\cap(\alpha\times\alpha)).$$

Using $\diamondsuit$, we can then pick an α such that in fact

$$(N_\alpha, \in, <_{\mathcal{H}_\theta}) \cong (X_\alpha, E_\alpha, <_\alpha).$$

Then clearly $(X_\alpha, E_\alpha, <_\alpha)$ codes a model, and by elementarity $\langle M_\beta \mid \beta < \omega_1\rangle$ and $\langle P^\beta \mid \beta < \omega_1\rangle$ belong to N_α. Moreover, we may assume that $\dot{Y} \in N_\alpha$.

Let $\bar{N_\alpha}$ be the transitive collapse of N_α, via collapsing function π_α. As all members of the transitive closure of M_β and P^β for $\beta < \alpha$ have rank $< \alpha$ (again, without loss of generality), they are mapped onto themselves by π_α. Also, since N_α contains all the ordinals $< \alpha$, it follows that the initial segments $\langle M_\beta \mid \beta < \alpha\rangle$ and $\langle P^\beta \mid \beta < \alpha\rangle$ are contained in the transitive collapse $\bar{N_\alpha}$. But then, we find ourselves in the situation of Case 1 (from the construction of the models), and it follows that $M_\alpha = \bar{N_\alpha}$.

As $\dot{Y} \in N_\alpha$, $Y \in N_\alpha[G]$. Then $Y = \pi_\alpha(Y)$ is in the transitive collapse of $N_\alpha[G]$ by π_α, which is equal to $M_\alpha[\pi_\alpha``G]$, the generic extension of M_α by the forcing $\pi_\alpha(\mathbb{P}\cap N_\alpha)$.

$$\begin{array}{ccccc} N_\alpha & \xrightarrow{\mathbb{P}\cap N_\alpha} & N_\alpha[G] & & \ni Y \\ \downarrow{\scriptstyle \pi_\alpha} & & \downarrow{\scriptstyle \pi_\alpha} & & \\ M_\alpha & \xrightarrow{\pi_\alpha(\mathbb{P}\cap N_\alpha)} & M_\alpha[\pi_\alpha``G] & & \ni \pi_\alpha(Y) = Y \end{array}$$

Since $\mathbb{P}\cap N_\alpha$ has the s.p.s.-property, so does $\pi_\alpha(\mathbb{P}\cap N_\alpha)$. Therefore, Y is indeed in a generic extension of an M_α via a forcing with the s.p.s.-property. □

Now we may apply condition (3) of Lemma 2.3 with $M = M_\alpha$ and $M' = M_\alpha[\pi_\alpha``G]$, and see that Y has infinite intersection with some member of $\bigcup_{\beta\leq\alpha} A_\beta$. Therefore, indeed, $\mathcal{A} = \bigcup_{\alpha<\aleph_1} A_\alpha$ is $\mathbb{P}$-indestructible.

It remains only to argue that $\mathcal{A}^{(L^*)^{\mathbb{P}}}$ has a Σ^1_2 definition. For this, first note that $\omega^\omega \cap L^* = \omega^\omega \cap L$, and that, by Lemma 3.2, we may assume that the $\diamondsuit$-sequence we chose in the beginning of the proof is Σ_1 definable over L_{ω_1}. Since the M_α's are chosen so that they are either defined from $\diamondsuit$ or chosen to be $<_L$-least, and the P^α's are also $<_L$-least, we can use a standard argument to show that the set B of (codes for) $\{P^\alpha \mid \alpha < \aleph_1\}$ is a Σ^1_2 set. Then, in $(L^*)^{\mathbb{P}}$, the mad family is given by the formula

$$x \in \mathcal{A} \iff \exists b \in B\, (x \in A_\alpha \text{ for } \alpha \text{ s.t. } b \text{ codes } P^\alpha).$$

Since "$x \in A_\alpha$ for α s.t. b codes P^α" is a recurisve computation, the above gives a Σ^1_2 definition of $\mathcal{A}$ in $(L^*)^{\mathbb{P}}$. This completes the proof of Lemma 4.1. □

With this we are almost done with the proof of the Main Theorem. All that remains to be done is forcing a Δ^1_3-definable well-order of the reals, together with $\mathfrak{b} = \mathfrak{c} = \aleph_3$, over L^* (i.e., the "coding stage"). For that, we will define a forcing iteration $\mathbb{P}_{\omega_3}$ following [7, Step 3], with only two essential differences:

1. for a.d. coding purposes, we will use the Laver-like almost disjoint coding from Lemma 2.4 as opposed to the standard a.d. coding, and
2. at stages where no coding is performed, we use a trivial version of the Laver-like coding (or use Hechler forcing).

This way, dominating reals are added cofinally often and the s.p.s.-property is preserved.

So, in L^*, fix a definable (e.g. closed) sequence $\vec{C} = \langle c_\zeta : \zeta < \omega_1\rangle$ of almost disjoint subsets of ω. This will be used for coding purposes. We will define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\gamma \mid \alpha \leq \omega_3, \gamma < \omega_3\rangle$ such that $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a σ-centered poset which has the s.p.s.-property. Every $\mathbb{Q}_\alpha$ is going to add a generic real whose $\mathbb{P}_\alpha$-name will be denoted by $\dot{u}_\alpha$, and just as in [7] we will have that $L^*[G_\alpha] \cap {}^\omega\omega = L^*[\langle \dot{u}_\xi^{G_\alpha} \mid \xi < \alpha\rangle] \cap {}^\omega\omega$ for every $\mathbb{P}_\alpha$-generic filter G_α.

This gives a canonical well-order of the reals in $L^*[G_\alpha]$, which depends only on the sequence $\langle \dot{u}_\xi^{G_\alpha} : \xi < \alpha\rangle$. The $\mathbb{P}_\alpha$-name for this well-order will be denoted by $\dot{<}_\alpha$. Additionally, we can make sure that for all $\alpha < \beta$ we have that $\mathbb{P}_\beta$ forces $\dot{<}_\alpha$ to be an initial segment of $\dot{<}_\beta$. Then if G is a $\mathbb{P}_{\omega_3}$-generic filter over L^*, $<^G := \bigcup\{\dot{<}_\alpha^G : \alpha < \omega_3\}$ will be the desired well-order of the reals.

We proceed with the recursive construction of $\mathbb{P}_{\omega_3}$. Along the construction we shall also define a sequence $\langle \dot{A}_\alpha \mid \alpha \in \mathrm{Lim}(\omega_3)\rangle$, where $\dot{A}_\alpha$ is a $\mathbb{P}_\alpha$-name for a subset of $[\alpha, \alpha+\omega)$. For every ν with $\omega_2 \le \nu < \omega_3$, fix a bijection $i_\nu : \{\langle\zeta,\xi\rangle \mid \zeta < \xi < \nu\} \xrightarrow{\sim} \mathrm{Lim}(\omega_2)$. If G_α is $\mathbb{P}_\alpha$-generic over L^*, $<_\alpha = \dot{<}_\alpha^{G_\alpha}$ and x, y are reals in $L^*[G_\alpha]$ such that $x <_\alpha y$, let $x * y := \{2n \mid n \in x\} \cup \{2n+1 \mid n \in y\}$ and $\Delta(x*y) := \{2n+2 \mid n \in x*y\} \cup \{2n+1 \mid n \notin x*y\}$.

Suppose $\mathbb{P}_\alpha$ has been defined and fix a $\mathbb{P}_\alpha$-generic filter G_α.

Suppose α is a limit ordinal. Write it in the form $\omega_2 \cdot \alpha' + \xi$, where $\xi < \omega_2$. If $\alpha' > 0$, let $i = i_{\mathrm{o.t.}(\dot{<}^{G_\alpha}_{\omega_2\cdot\alpha'})}$ and $\langle \xi_0, \xi_1\rangle = i^{-1}(\xi)$. Let $A_\alpha := \dot{A}_\alpha^{G_\alpha}$ be the set $\alpha + (\omega \backslash \Delta(x_{\xi_0} * x_{\xi_1}))$, where x_ζ is the ζ-th real in $L[G_{\omega_2\cdot\alpha'}] \cap [\omega]^\omega$ according to the well-order $\dot{<}^{G_\alpha}_{\omega_2\cdot\alpha'}$ (here $G_{\omega_2\cdot\alpha'} = G_\alpha \cap \mathbb{P}_{\omega_2\cdot\alpha'}$).

Then, we define $\mathbb{Q}_\alpha$ as follows: $\mathbb{Q}_\alpha$ is the finite support iteration $\langle \mathbb{P}^n_\alpha, \dot{\mathbb{Q}}^m_\alpha \mid n \le \omega, m < \omega\rangle$, where

- **Case 1:** if $m \in \Delta(x_{\xi_0} * x_{\xi_1})$ then $\Vdash_m$ "$\dot{\mathbb{Q}}^m_\alpha$ is the Laver-like a.d. coding partial order $\mathbb{L}_{Y_{\alpha+m}}(\vec{C})$ from Lemma 2.4", where $\vec{C}$ is the a.d. sequence fixed at the beginning, and $Y_{\alpha+m}$ is the subset of ω_1 whose characteristic function was added by $\mathbb{P}^2_{\alpha+m}$ (see Section 2).
- **Case 2:** if $m \notin \Delta(x_{\xi_0} * x_{\xi_1})$ then $\Vdash_m \dot{\mathbb{Q}}^m_\alpha$ is the trivial poset.

Let u^m_α be the generic real added by $\mathbb{Q}^m_\alpha$ in the first case, and the constant 0 function in the second case. Let u_α be a real encoding the u^m_α's for all $m \in \omega$.

If $\alpha < \omega_2$ or α is a successor, let $\mathbb{Q}_\alpha$ be again the Laver-like forcing $\mathbb{L}_{\omega_1}(\vec{C})$, or Hechler forcing (or any other σ-centered forcing that satisfies the s.p.s. property). Notice that what happens at these stages is irrelevant for the purpose of "decoding" the Δ^1_3-well-order.

With this the inductive definition of our finite support iteration $\mathbb{P}_{\omega_3}$ is complete—for more details, we refer the reader to [7]. To complete the proof, first notice that since the sets $\Delta(x, y)$ are always non-empty, Case 1 occurrs cofinally often in the iteration, and therefore dominating reals are added cofinally often. It follows that in $(L^*)^{\mathbb{P}_{\omega_3}}$ we have $\mathfrak{b} = \mathfrak{c} = \aleph_3$. To show that in $(L^*)^{\mathbb{P}_{\omega_3}}$ there is a Δ^1_3-definable well-order of the reals, we follow the arguments of [7]. Notice that a version of [7, Lemma 3] certainly goes through in our context (in fact it is even easier to prove). Consequently, Lemmas 4 and 5 from [7] hold, and the Σ^1_3 formula defining the well-order can be read off from the statements of these Lemmas.

Finally, note that all the forcing posets in the construction of $\mathbb{P}_{\omega_3}$ (including the trivial ones) satisfy the s.p.s.-property, so by [2, Proposition 3.10], the entire finite support iteration does, as well. Thus we can apply Lemma 4.1

and obtain a Σ^1_2-definable mad family in $(L^*)^{\mathbb{P}_{\omega_3}}$, and, by Theorem 1.3, also a Π^1_1 mad family.

5 Open Questions

In the introduction, we mentioned that there is no problem to modify our proof so that it works for $\mathfrak{b} = \mathfrak{c} = \aleph_2$, and a natural open question is whether the same holds for $\mathfrak{b} = \mathfrak{c} = \kappa$ for all regular uncountable κ. We conjecture that the answer is positive, but some work needs to be done on the coding mechanism to make sure it works for larger values of the continuum.

Another question one may ask is whether the existence of a Π^1_1 mad family and a Δ^1_3 well-order is consistent with other values of the cardinal characteristics $\mathfrak{b}, \mathfrak{a}$ and $\mathfrak{s}$. For example, is it consistent with $\mathfrak{b} < \mathfrak{c}$ or even $\mathfrak{b} < \mathfrak{a}$?

6 Acknowledgments

The authors acknowledge financial support from the Austrian Science Fund (FWF) under project numbers P22430-N13 (Vera Fischer and Sy Friedman), M1365-N13 (Vera Fischer) and P23316-N13 (Sy Friedman and Yurii Khomskii), and partial support by the European Science Foundation (ESF) for the activity "New Frontiers of Infinity: Mathematical, Philosophical and Computational Prospects" (INFTY) (Yurii Khomskii).

The authors would like to thank Carlos Martinez-Ranero for drawing their attention to [3].

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