Cichoń's diagram, regularity properties and $\mathbf{\Delta}^1_3$ sets of reals.

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Abstract

We study regularity properties related to Cohen, random, Laver, Miller and Sacks forcing, for sets of real numbers on the $\mathbf{\Delta}^1_3$ level of the projective hieararchy. For $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ sets, the relationships between these properties follows the pattern of the well-known Cichoń diagram for cardinal characteristics of the continuum. It is known that assuming suitable large cardinals, the same relationships lift to higher projective levels, but the questions become more challenging without such assumptions. Consequently, all our results are proved on the basis of ZFC alone or ZFC with an inaccessible cardinal. We also prove partial results concerning $\mathbf{\Sigma}^1_3$ and $\mathbf{\Delta}^1_4$ sets.

1 Introduction

The study of regularity properties in descriptive set theory is closely related to cardinal characteristics of the continuum. By well-known results of Solovay, Judah and Shelah, the statement "all $\mathbf{\Sigma}^1_2$ sets of reals are Lebesgue measurable" is equivalent to "for every $r \in \omega^\omega$, the set of random reals over $L[r]$ has measure one", and the statement "all $\mathbf{\Delta}^1_2$ sets of reals are Lebesgue measurable" is equivalent to "for every $r \in \omega^\omega$, there is a random real over $L[r]$"; analogous results hold for the Baire property and Cohen reals. These characterizations link the statements about projective regularity with the covering and additivity numbers of the meager and null ideals on the reals. Likewise, Brendle and Löwe [8] uncovered a link between the regularity properties naturally connected to Laver-, Miller- and Sacks-forcing for $\mathbf{\Sigma}^1_2$ and $\mathbf{\Delta}^1_2$ sets, and the cardinal invariants $\mathfrak{b}$ (the bounding number), $\mathfrak{d}$ (the dominating number) and the size of the continuum.

Thus, if we restrict attention to the second projective level, a very clear picture emerges, in which the relationships between the various regularity statements follow the familiar pattern of Cichoń's diagram (see Figure 1 in Section 2). Other, more exotic, regularity properties have also been extensively studied on the second level, with some important contributions being [23, 8, 6, 9, 7]. See also [32] for a very detailed and self-contained survey. An abstract approach has been proposed by Ikegami in [24] and developed further in the PhD theses of Laguzzi [36] and of the third author [35].

Far less is known concerning sets higher up in the projective hierarchy, even at the $\mathbf{\Sigma}^1_3$ and $\mathbf{\Delta}^1_3$ levels. Concerning such questions, there are two, somewhat divergent, methods of approach. According to one of them, adopted e.g. by Ikegami in [24], Judah and Spinas in [31] and a few others, one assumes the existence of certain large cardinals, which imply that all the essential results from the second level lift almost verbatim to higher levels (for the third level, this requires the existence of sharps for sets of ordinals). Although this approach is interesting and certainly worthy of further investigation, it is not the approach we will take in this paper, for reasons that shall be explained in the next section. Here, all results will be proved on the basis of ZFC alone or ZFC with an inaccessible; indeed, we will put special emphasis on eliminating the inaccessible wherever possible (notice that the statement "all $\mathbf{\Sigma}^1_3$ sets are Lebesgue measurable" already implies an inaccessible in L by [40]).

Some work in this direction, most of it contained in Chapter 9 of [3], has been carried out by Judah, Shelah, Bagaria and others in the eighties and early nineties, and our methods are related to the ones used there. On the other hand, we have more modern means at our disposal, particularly the theory of "non-elementary proper forcing" (in our case, "Suslin and Suslin$^+$ proper forcing") developed by Judah, Shelah, Goldstern and Kellner, and a result of René David [10] about the existence of a model of set theory in which ω_1 is inaccessible in $L[r]$ for all reals r, but there exists a $\mathbf{\Sigma}^1_3$-good wellorder of the reals. Using these methods, we will provide a complete solution to the situation on the $\mathbf{\Delta}^1_3$-level. Although our emphasis will be on the regularity properties corresponding to the cardinal invariants appearing in Cichoń's diagram (i.e., the regularity properties connected to Cohen, random, Laver, Miller and Sacks forcing), our methods are sufficiently general and certainly have many more applications regarding questions of a similar nature.

The paper is structured as follows: in Section 2 we introduce the relevant definitions, summarize known results on the second level and provide the motivation for the research carried out in the rest of the paper. In Section 3 we recall the basic properties of Suslin and Suslin$^+$ proper forcing, proving several important technical results which may be interesting in their own right and have applications other than those considered in this paper. In the crucial Section 4 we develop several methods for obtaining regularity for $\mathbf{\Delta}^1_3$ sets of reals in a "minimal" way, using various iterated forcing techniques. In Section 5 we use these methods to separate regularity properties on the $\mathbf{\Delta}^1_3$-level. In Section 6 we briefly consider two additional regularity properties that have received a lot of attention is set theory, and in Section 7 we deal with some results concerning $\mathbf{\Sigma}^1_3$ and $\mathbf{\Delta}^1_4$ sets. Section 8 closes with some open questions.

2 Regularity properties and Cichoń's diagram

2.1 Definitions

We assume that the reader is familiar with the standard definitions of the *Baire property*, *Lebesgue measure*, the ideal $\mathcal{M}$ of *meager* sets and $\mathcal{N}$ of *measure-null* sets, as well as the definitions of *Cohen*, *random*, *Laver*, *Miller* and *Sacks forcing*. Following standard practice, we denote these forcing notions with the letters $\mathbb{C}, \mathbb{B}, \mathbb{L}, \mathbb{M}$ and $\mathbb{S}$. If T is a tree on $\omega^{<\omega}$ or $2^{<\omega}$ then $[T]$ denotes the set of branches through T, and $[t]$ denotes the basic open set for $t \in \omega^{<\omega}$ or $2^{<\omega}$.

Definition 2.1. A set $A \subseteq \omega^\omega$ is

- *Laver-measurable* if $\forall T \in \mathbb{L}\, \exists S \in \mathbb{L}$ s.t. $S \leq T$ and $([S] \subseteq A$ or $[S] \cap A = \varnothing)$.
- *Miller-measurable* if $\forall T \in \mathbb{M} \exists S \in \mathbb{M}$ s.t. $S \leq T$ and $([S] \subseteq A$ or $[S] \cap A = \varnothing)$.

  A set $A \subseteq 2^\omega$ is

- *Sacks-measurable* if $\forall T \in \mathbb{S}\, \exists S \in \mathbb{S}$ s.t. $S \leq T$ and $([S] \subseteq A$ or $[S] \cap A = \varnothing)$.

Sacks-measurability is also known under the term *Marczewski-measurability.* Although contemporary interest in properties such as the ones above is often forcing-related, it is interesting to note that among Polish mathematicians, there had been a considerable interest in them long before the advent of forcing, see e.g. [44].

Both Lebesgue measure and the Baire property can be represented in the style of Definition 2.1, using the following well-known characterizations:

1. A subset A of ω^ω or 2^ω is Lebesgue-measurable iff every closed set C of positive measure has a closed subset $C' \subseteq C$ of positive measure such that $C' \subseteq A$ or $C' \cap A = \varnothing$.

2. A subset A of ω^ω or 2^ω has the Baire property iff every basic open set $[t]$ has a basic open subset $[s] \subseteq [t]$ such that $[s] \setminus A$ is meager or $[s] \cap A$ is meager. Moreover, this holds iff every G_δ non-meager set X has a G_δ non-meager subset $Y \subseteq X$ such that $Y \subseteq A$ or $Y \cap A = \varnothing$.

If we choose to represent random forcing by the partial order of closed sets of positive measure, and Cohen forcing by G_δ (or Borel) non-meager sets, we obtain an exact equivalence between the two classical properties on one hand, and $\mathbb{B}$- and $\mathbb{C}$-measurability in the sense analogous to Definition 2.1 on the other hand. Therefore, we will frequently refer to the Baire property and Lebesgue measure as "$\mathbb{C}$-" and "$\mathbb{B}$-measurability", respectively.

Notation 2.2. If $\mathbf{\Gamma}$ is a class of sets (e.g. a projective class), we will use the notation "$\mathbf{\Gamma}(\mathbb{P})$" to abbreviate the statement "all sets of complexity $\mathbf{\Gamma}$ are $\mathbb{P}$-measurable.", with $\mathbb{P}$ ranging over one of the forcing notions considered above.

2.2 The second level

While ZFC proves that analytic sets are $\mathbb{P}$-measurable for all $\mathbb{P}$ as above, statements such as $\boldsymbol{\Sigma}^1_2(\mathbb{P})$ and $\boldsymbol{\Delta}^1_2(\mathbb{P})$ are independent of ZFC. The following results of Solovay [42], Judah-Shelah [23] and Brendle-Löwe [8] provide an exact characterization of regularity statements for $\boldsymbol{\Sigma}^1_2$ and $\boldsymbol{\Delta}^1_2$ sets of reals.

Theorem 2.3 (Solovay, 1970)**.**

1. $\boldsymbol{\Sigma}^1_2(\mathbb{B}) \iff \forall r \, \{x \mid x \text{ is not random over } L[r]\} \in \mathcal{N}$.
2. $\boldsymbol{\Sigma}^1_2(\mathbb{C}) \iff \forall r \, \{x \mid x \text{ is not Cohen over } L[r]\} \in \mathcal{M}$.

Theorem 2.4 (Judah-Shelah, 1989)**.**

1. $\boldsymbol{\Delta}^1_2(\mathbb{B}) \iff \forall r \, \exists x \, (x \text{ is random over } L[r])$.
2. $\boldsymbol{\Delta}^1_2(\mathbb{C}) \iff \forall r \, \exists x \, (x \text{ is Cohen over } L[r])$.

Theorem 2.5 (Brendle-Löwe, 1999)**.**

1. $\boldsymbol{\Sigma}^1_2(\mathbb{L}) \iff \boldsymbol{\Delta}^1_2(\mathbb{L}) \iff \forall r \, \exists x \, (x \text{ is dominating over } L[r])$.
2. $\boldsymbol{\Sigma}^1_2(\mathbb{M}) \iff \boldsymbol{\Delta}^1_2(\mathbb{M}) \iff \forall r \, \exists x \, (x \text{ is unbounded over } L[r])$.
3. $\boldsymbol{\Sigma}^1_2(\mathbb{S}) \iff \boldsymbol{\Delta}^1_2(\mathbb{S}) \iff \forall r \, \exists x \, (x \notin L[r])$.

These three theorems make it possible to compare the strength of various hypotheses of the form $\boldsymbol{\Sigma}^1_2(\mathbb{P})$ and $\boldsymbol{\Delta}^1_2(\mathbb{P})$ with one another. Notice that the right-hand-side statements of Theorem 2.3 are naturally related to the cardinal numbers $\mathrm{add}(\mathcal{N})$ and $\mathrm{add}(\mathcal{M})$; the right-hand-side statement of Theorem 2.4 are related to $\mathrm{cov}(\mathcal{N})$ and $\mathrm{cov}(\mathcal{M})$; and those of Theorem 2.5 to $\mathfrak{b}$, $\mathfrak{d}$ and $2^{\aleph_0}$. So it is not surprising that the relationship between the regularity hypotheses follows a pattern familiar from (part of) the Cichoń diagram—see Figure 1.

The interpretation of this diagram is as usual: every implication appearing on it is provable in ZFC, as well as the additional implication $\boldsymbol{\Delta}^1_2(\mathbb{L})+\boldsymbol{\Delta}^1_2(\mathbb{C}) \Rightarrow \boldsymbol{\Sigma}^1_2(\mathbb{C})$ (the counterpart to the cardinal equation $\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$ established by John Truss [45]). Any other implication is *not* provable, i.e., any constellation of true/false-assignments to the above statements not contradicting the diagram, is actually consistent with ZFC. We call such an implication diagram "complete". The above facts are well-known, and can be proved by iterating the right type of forcing notions over L and using the fact that certain types of reals are, or are not, added by the iteration, thus forcing the right-hand-side statements of Theorems 2.3, 2.4 and 2.5 to be true or false. This is in perfect analogy to the proofs of the corresponding cardinal inequalities, which can be found e.g. in [3, Chapter 7].

Notice that the statement $\forall r(\omega_1^{L[r]} < \omega_1)$ is a little bit special, since it is the only one that requires the strength of an inaccessible; nevertheless, it is a natural property in this setting because:

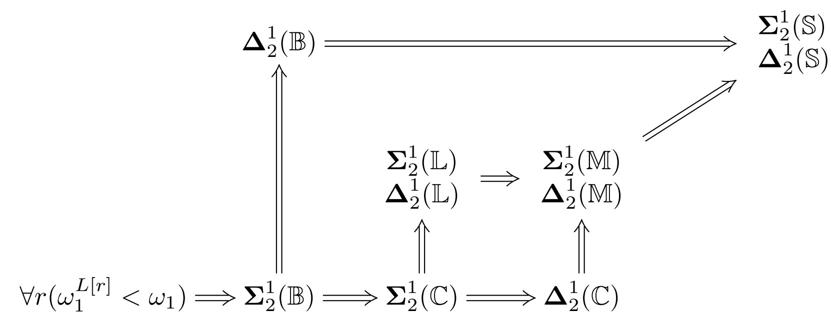

Figure 1: A complete diagram of implications for $\mathbf{\Sigma}^1_2$ and $\mathbf{\Delta}^1_2$ sets of reals.

(a) it plays the same role as $\aleph_1$ does in the standard Cichoń diagram for cardinal characteristics, and

(b) it is equivalent to a number of projective regularity statements, most notably "all $\mathbf{\Sigma}^1_2/\mathbf{\Pi}^1_1$ sets have the perfect set property".

The correspondence between regularity hypotheses on the second level, transcendence over L and cardinal characteristics of the continuum is summarized in Table 1 below.

Reg. hypothesis	Transcendence over $L[r]$	Cardinal char.
$\forall r(\omega_1^{L[r]} < \omega_1)$	"making ground model reals countable"	$\aleph_1$
$\mathbf{\Sigma}^1_2(\mathbb{B})$	measure-one many random reals	$\text{add}(\mathcal{N})$
$\mathbf{\Delta}^1_2(\mathbb{B})$	random reals	$\text{cov}(\mathcal{N})$
$\mathbf{\Sigma}^1_2(\mathbb{C})$	co-meager many Cohen reals	$\text{add}(\mathcal{M})$
$\mathbf{\Delta}^1_2(\mathbb{C})$	Cohen reals	$\text{cov}(\mathcal{M})$
$\mathbf{\Delta}^1_2(\mathbb{L})$ / $\mathbf{\Sigma}^1_2(\mathbb{L})$	dominating reals	$\mathfrak{b}$
$\mathbf{\Delta}^1_2(\mathbb{M})$ / $\mathbf{\Sigma}^1_2(\mathbb{M})$	unbounded reals	$\mathfrak{d}$
$\mathbf{\Delta}^1_2(\mathbb{S})$ / $\mathbf{\Sigma}^1_2(\mathbb{S})$	new reals	$2^{\aleph_0}$

Table 1: Correspondence between regularity, transcendence and cardinal characteristic.

2.3 Beyond the second level

When looking higher up in the projective hierarchy and attempting to generalize the theory to statements like $\mathbf{\Sigma}^1_n(\mathbb{P})$ and $\mathbf{\Delta}^1_n(\mathbb{Q})$, for $n \geq 3$, we are faced with two distinct methods of approach, as mentioned in the introduction. For example, if $L^\#$ denotes the least inner model closed under sharps for sets of ordinals, Theorems 2.3, 2.4 and 2.5 can be lifted to the next level, so in set-generic extensions of $L^{\#}$ we obtain characterizations of $\mathbf{\Sigma}^1_3(\mathbb{P})$ and $\mathbf{\Delta}^1_3(\mathbb{P})$ in terms of transcendence properties over $L^{\#}$. An immediate consequence is that all the properties of the diagram from Figure 1 lift to the third projective level as well. For more on this approach, see the work of Ikegami [24, Section 5]. Judah and Spinas [31] also proved results such as: if V is a canonical model with n Woodin cardinals and a measurable above them, then there is a forcing extension in which $\mathbf{\Delta}^1_{n+4}(\mathbb{B})$ holds but $\mathbf{\Delta}^1_{n+4}(\mathbb{C})$ fails.

In this paper, we do not adopt the "large cardinal approach", for the following reasons:

1. As the consistency of "for all $\mathbb{P}$ and $n < \omega$, $\mathbf{\Sigma}^1_n(\mathbb{P})$ holds" is just an inaccessible (it is true in the Solovay model), it seems unnatural to require stronger hypotheses to prove more subtle statements about $\mathbf{\Sigma}^1_n(\mathbb{P})$ or $\mathbf{\Delta}^1_n(\mathbb{P})$ for low values of n (this view has been expressed by Bagaria, Judah, Shelah and others in the past).

2. Assuming *too* strong large cardinals (for example, enough to yield Projective Determinacy) may trivialize the question. So, for this approach to work properly one must assume *exactly the right* amount of large cardinal strength, which is, arguably, a somewhat artificial requirement.

3. Without large cardinal assumptions, one can obtain results that are not direct analogues of the second level results. In recent work of Friedman and Schrittesser [16], a model for $\mathbf{Proj}(\mathbb{B})+\neg\mathbf{\Delta}^1_3(\mathbb{C})$ was constructedt ("**Proj**" stands for the class of all projective sets). In particular, this showed that the counterpart to the classical Bartoszyński-Raisonnier-Stern implication "$\mathbf{\Sigma}^1_n(\mathbb{B}) \Rightarrow \mathbf{\Sigma}^1_n(\mathbb{C})$" fails to lift to higher levels, for all $n \geq 3$ (on the other hand, the existence of a measurable implies $\mathbf{\Sigma}^1_3(\mathbb{B}) \Rightarrow \mathbf{\Sigma}^1_3(\mathbb{C})$). Other "non-liftings" of implications will follow from our results as well, for example that $\mathbf{\Delta}^1_n(\mathbb{L}) + \mathbf{\Delta}^1_n(\mathbb{C}) \Rightarrow \mathbf{\Sigma}^1_n(\mathbb{C})$ (the analogue of the Truss-implication) consistently fails for $n = 3$ and $n = 4$, see Theorem 7.12. In light of this, it seems more interesting to study such questions in ZFC or at most ZFC with an inaccessible.

So, if we must forgo large cardinal assumptions beyond an inaccessible, we must also forgo beautiful characterization theorems like Theorem 2.3, 2.4 and 2.5. But then, is there anything at all we can say about the relationship between the five regularity properties? Fortunately, a number of simple implications can be proved by straightforward ZFC-arguments. First, an important observation:

Observation 2.6 (Brendle-Löwe)**.** Let $\mathbb{P} \in \{\mathbb{B}, \mathbb{L}, \mathbb{M}, \mathbb{S}\}$. For any tree $T \in \mathbb{P}$, there exists a natural homeomorphism φ_T between $[T]$ and the entire space (ω^ω or 2^ω), which preserves the property of "being a $\mathbb{P}$-condition". From this it follows that if $\mathbf{\Gamma}$ is a class of sets closed under continuous preimages, and we are only interested in the statement $\mathbf{\Gamma}(\mathbb{P})$, then we may safely drop the "below any $\mathbb{P}$-condition"-clause from the definition of $\mathbb{P}$-measurability, and simply say that a set A is $\mathbb{P}$-measurable if and only if there exists a $T \in \mathbb{P}$ such that $[T] \subseteq A$ or

$[T] \cap A = \varnothing$. Similarly, A is $\mathbb{C}$-measurable if and only if there is a G_δ non-meager set X such that $X \subseteq A$ or $X \cap A = \varnothing$.

Lemma 2.7 (Brendle-Löwe)**.** *Let* $\mathbf{\Gamma}$ *be a class of sets closed under continuous pre-images. Then the following implications hold in ZFC:*

1. $\mathbf{\Gamma}(\mathbb{L}) \Rightarrow \mathbf{\Gamma}(\mathbb{M}) \Rightarrow \mathbf{\Gamma}(\mathbb{S})$.
2. $\mathbf{\Gamma}(\mathbb{C}) \Rightarrow \mathbf{\Gamma}(\mathbb{M})$.
3. $\mathbf{\Gamma}(\mathbb{B}) \Rightarrow \mathbf{\Gamma}(\mathbb{S})$.

Proof. In view of the previous observation, proving $\mathbf{\Gamma}(\mathbb{P}) \Rightarrow \mathbf{\Gamma}(\mathbb{Q})$ amounts to finding a $\mathbb{Q}$-object below any $\mathbb{P}$-object. For the first implication, note that a Laver tree is a Miller tree, a Miller tree is a perfect tree in ω^ω, and the imagine of this perfect tree under the natural homeomorphism between ω^ω and a dense G_δ subset of 2^ω, is an uncountable G_δ subset of 2^ω which, by the perfect set theorem, contains the branches of a Sacks tree.

For the second implication, note that a G_δ non-meager set is comeager in a basic open set. It is not hard to inductively construct a Miller tree whose branches are completely contained inside a set that is comeager in a basic open set.

Finally, every closed set of positive measure clearly contains a perfect subset. □

Summarizing the above, we obtain a different implicaiton diagrams for the same regularity properties on the $\mathbf{\Sigma}^1_n$ and $\mathbf{\Delta}^1_n$ level, for $n \geq 3$, see Figure 2. Note, however, that unlike Figure 1, this is not a "complete" diagram, in the sense that it only shows the implications we know to exist so far, but it does not claim that no additional implications exist. Also, notice that the analogue of $\forall r(\omega_1^{L[r]} < \omega_1)$ is missing from the diagram—it is not clear which hypothesis should take its place.

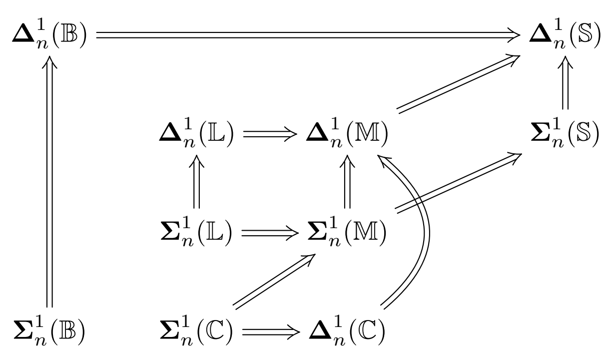

Figure 2: An incomplete diagram of implications for $\mathbf{\Sigma}^1_n$ and $\mathbf{\Delta}^1_n$ sets of reals.

The long-term goal is to "complete" this diagram on the third level, and potentially on all levels $n \geq 3$ (i.e., to find all possible implications and prove that all other implications are consistently false). There are still many obstacles to this goal. However, if we restrict attention exclusively to the $\mathbf{\Delta}^1_3$ sets, we obtain a much simpler diagram (see Figure 3). In Section 5 we show that that diagram is indeed complete, by constructing models for every combination of "true"/"false"-assignments consistent with the diagram, in ZFC or ZFC with an inaccessible. Partial results related to levels above $\mathbf{\Delta}^1_3$ will be discussed in Section 7.

We should mention that results concerning the Baire propety and Lebesgue measurability were known prior to our work. The consistency of $\mathbf{\Delta}^1_3(\mathbb{C}) + \neg\mathbf{\Delta}^1_3(\mathbb{B})$, for example, follows from [27], and the consistency of the converse, $\mathbf{\Delta}^1_3(\mathbb{B}) + \neg\mathbf{\Delta}^1_3(\mathbb{C})$, was first proved by Bagaria in [28] and later (using different methods) by Bagaria and Woodin in [2]. The consistency of $\mathbf{\Sigma}^1_3(\mathbb{B}) + \neg\mathbf{\Delta}^1_3(\mathbb{C})$ had remained open for a long time, until it became a corollary of the much stronger theorem of [16]. To our knowledge, no study of the properties $\mathbb{L}$, $\mathbb{M}$ and $\mathbb{S}$ on higher levels has been carried out so far.

On the technical side, our proofs will involve Suslin and Suslin$^+$ proper forcings, a special case of the general theory of "non-elementary proper forcing" developed by Shelah, cf. [41].

3 Suslin and Suslin$^+$ proper forcing

The theory of Suslin ccc forcings is well-understood, and a detailed summary can be found in [3, Section 3.6]. In the context of forcing notions that are proper but not ccc, there is a closely related concept, developed, among others, by Judah and Shelah in [22], Goldstern in [19, 18], Shelah in [41] and Kellner in [33, 34]. In this section we will give a brief overview of some essential properties of Suslin and Suslin$^+$ proper forcing, and prove some results that will be crucial for the techniques in our paper.

3.1 Basic concepts

The main idea is to replace countable elementary submodels $M \prec \mathcal{H}_\kappa$ for sufficiently large κ in the definition of "proper forcing" by countable transitive (not necessarily collapses of elementary) models of (a sufficient fragment of) ZFC. For that to make sense, the forcing notions need to be definable.

Definition 3.1. Let $\mathbb{P}$ be a forcing partial order whose conditions are (or can be coded by) reals. Assume that $\mathbb{P}$, $\leq_\mathbb{P}$ and $\perp_\mathbb{P}$ are definable by projective formulas with a parameter $a \in \omega^\omega$. Let ZFC* denote some (unspecified) sufficiently large finite fragment of ZFC, and let M be a countable transitive model of ZFC* containing the parameter a. Then $\mathbb{P}^M$, $<^M_\mathbb{P}$ and $\perp^M_\mathbb{P}$ refer to the forcing notion re-interpreted in M. A condition $q \in \mathbb{P}$ is called $(M, \mathbb{P})$-generic if (in V) $q \Vdash$ "$\dot{G} \cap \mathbb{P}^M$ is a $\mathbb{P}^M$-generic filter over M".

Following the terminology introduced by Shelah, countable models of ZFC* which contain the defining parameters will be called "candidates".

Definition 3.2. Let $(\mathbb{P}, \leq_{\mathbb{P}}, \perp_{\mathbb{P}})$ be as above. We say that $\mathbb{P}$ is *proper-for-candidates* if for all candidates M containing the defining parameter of $\mathbb{P}$, and every $p \in \mathbb{P}^M$, there exists a $q \leq p$ which is $(M, \mathbb{P})$-generic.

Some authors call this property "strongly proper", although we will stick to the above terminology in order to avoid confusion with other interpretations of the term "strongly proper".

Note that if $M \prec \mathcal{H}_\kappa$ is a countable elementary submodel of a sufficiently large $\mathcal{H}_\kappa$ such that $\mathcal{H}_\kappa \models \text{ZFC}^*$ and contains all relevant parameters, then a condition q is $(\overline{M}, \mathbb{P})$-generic in the above sense if and only if it is $(M, \mathbb{P})$-generic in the usual sense (with $\mathbb{P}^M = \mathbb{P} \cap M$). Hence, properness-for-candidates implies ordinary properness.

Usually, properness-for-candidates is coupled with an absoluteness requirement on the definition of the partial order.

Definition 3.3. A forcing $\mathbb{P}$ is *Suslin proper* if $\mathbb{P}$, $\leq_{\mathbb{P}}$ and $\perp_{\mathbb{P}}$ are $\mathbf{\Sigma}^1_1$-relations, and $\mathbb{P}$ is proper-for-candidates.

If $\mathbb{P}$ is Suslin proper, then $\mathbb{P}^M = \mathbb{P} \cap M$, $\leq^M_{\mathbb{P}} = \leq_{\mathbb{P}} \cap M^2$ and $\perp^M_{\mathbb{P}} = \perp_{\mathbb{P}} \cap M^2$ by $\mathbf{\Sigma}^1_1$-absoluteness. Moreover, the statement "$\{p_i \mid i < \omega\}$ is predense below q" is $\mathbf{\Pi}^1_1$ and hence absolute between candidates M and V. Clearly, all Suslin ccc partial orders (i.e., all Suslin partial orders having the ccc) are Suslin proper, and there are some well-known examples of non-ccc forcings that are Suslin proper—most notably Mathias forcing. However, many standard forcing notions (e.g., Sacks, Miller and Laver forcing) are not quite Suslin proper, because $\perp_{\mathbb{P}}$ fails to be a $\mathbf{\Sigma}^1_1$ relation (it is then only $\mathbf{\Pi}^1_1$). To fix this problem, an alternative notion was proposed by Shelah and Goldstern:

Definition 3.4. A forcing $\mathbb{P}$ is *Suslin$^+$ proper* if

1. $\mathbb{P}$ and $\leq_{\mathbb{P}}$ are $\mathbf{\Sigma}^1_1$,
2. there is a $\mathbf{\Sigma}^1_2$, $(\omega + 1)$-place relation $\text{epd}(p_0, p_1, \ldots, q)$ ("effectively predense") such that if $\text{epd}(p_0, p_1, \ldots, q)$ holds for $p_i, q \in \mathbb{P}$, then $\{p_i \mid i < \omega\}$ is predense below q, and
3. for every candidate M containing all relevant parameters, and all $p \in \mathbb{P}^M$, there is a $q \leq p$ such that for every $D \in M$ which is $\mathbb{P}^M$-dense, there exists an enumeration $\{d_i \mid i < \omega\} \subseteq D$ such that $\text{epd}(d_0, d_1, \ldots, q)$ holds. In this case we say that q is an *effective* $(M, \mathbb{P})$-generic condition, and we call this property *effective*-properness-for-candidates.

So Suslin properness implies Suslin$^+$ properness, which in turn implies properness. A sufficient condition for a forcing to be Suslin$^+$ proper is an effective version of Axiom A, where the amalgamation makes sure that epd is defined in a $\mathbf{\Sigma}^1_2$-way. All standard definable tree-like forcings which are known to be proper are in fact Suslin$^+$ proper. A good exposition of this phenomenon can be found in Kellner's papers [33, 34].

Remark 3.5. In [18, Remark 1.7] it was shown that if $d \in \omega^\omega$ is a code for an analytic set, canonically coding $(\mathbb{P}, \leq_\mathbb{P}, \perp_\mathbb{P}))$, then the statement "$d$ codes a Suslin proper forcing" is a $\mathbf{\Pi}^1_3$ statement. The same holds for Suslin$^+$ proper forcing, i.e., if d is a code for a $\mathbf{\Sigma}^1_2$-set canonically coding $(\mathbb{P}, \leq_\mathbb{P})$ as well as the relation epd, then "d codes a Suslin$^+$ proper forcing" iff

$$\begin{gathered}\leq_\mathbb{P} \text{ is a partial order, and} \\ \forall^1 M \, [M \text{ countable, transitive, } M \models \mathrm{ZFC}^*, d \in M \to \\ \forall^0 p \in \mathbb{P}^M \;\; \exists^1 q \leq_\mathbb{P} p \quad \text{s.t.} \quad \forall^0 D \in M (M \models \text{“}D \text{ is dense”} \to \\ \exists^1 \{d_i \mid i < \omega\} \subseteq D \text{ s.t. } \mathrm{epd}(d_0, d_1, \ldots, q))]\end{gathered}$$

(where $\forall^0$ and $\exists^0$ refers to natural number quantifiers and $\forall^1$ and $\exists^1$ to real number quantifiers.) As countable, transitive models M can be coded by well-founded relations E on ω, it is not hard to verify that the above statement is $\mathbf{\Pi}^1_3$. In particular, if $\mathbb{P}$ is a Suslin (Suslin$^+$) proper forcing then $N \models$ "$\mathbb{P}$ is a Suslin (Suslin$^+$) proper forcing" for any inner model N with $\omega_1 \subseteq N$, by downwards $\mathbf{\Pi}^1_3$-absoluteness.

Next, we want to look at the complexity of the forcing relation $\Vdash_\mathbb{P}$. First, let us fix the following terminology:

Definition 3.6. Let $\mathbb{P}$ be a forcing notion. We say that τ is a *countable $\mathbb{P}$-name for a real* if it is a countable set of pairs $(\check{n}, p)$, where $n \in \omega$ and $p \in \mathbb{P}$.

In the above definition we think of reals as subsets of ω (or members of 2^ω), and if τ is of the above form and G a generic filter, then we think of $\tau[G]$ as the set $\{n \mid \exists p \in G \, ((\check{n}, p) \in \tau)\} \subseteq \omega$ (or the corresponding function in 2^ω).

Although not every name for a real is countable, if $\mathbb{P}$ is proper then for every $\mathbb{P}$-name for a real σ and $p \in \mathbb{P}$ there exists $q \leq p$ and a countable $\mathbb{P}$-name τ for a real such that $q \Vdash \tau = \sigma$. If conditions of $\mathbb{P}$ are reals, each such countable name can be canonically coded by a real. Moreover, if $\mathbb{P}$ is $\mathbf{\Sigma}^1_1$ then the statement "x codes a countable name for a real" is $\mathbf{\Sigma}^1_1$. We will frequently identify countable $\mathbb{P}$-names for reals with the reals coding them.

The following lemma generalizes [29, Theorem 2.1], and is crucial for computing the complexity of the forcing relation. In its formulation, "$p \Vdash \theta(\tau)$" is to be understood as a formula with real variables p and τ (actually the reals coding them)

Lemma 3.7. *Let $\mathbb{P}$ be Suslin$^+$ proper, $p \in \mathbb{P}$ and τ a countable $\mathbb{P}$-name for a real. Then for all $n \geq 2$:*

1. *If θ is $\mathbf{\Pi}^1_n$ then "$p \Vdash \theta(\tau)$" is $\mathbf{\Pi}^1_n$.*
2. *If θ is $\mathbf{\Sigma}^1_n$ then "$p \Vdash \theta(\tau)$" is $\mathbf{\Pi}^1_{n+1}$.*

Proof. The proof is by induction on the complexity of θ, with $\mathbf{\Pi}^1_2$ being the base case. So first, assume θ is $\mathbf{\Pi}^1_2$.

Claim. The following are equivalent:

1. $p \Vdash \theta(\tau)$,
2. for all candidates M containing τ, p, and any parameters appearing in the definition of $\mathbb{P}$ or θ, we have $M \models p \Vdash \theta(\tau)$.

As candidates are coded by well-founded relations E on ω, the above equivalence gives us a $\mathbf{\Pi}^1_2$-definition of "$p \Vdash \theta(\tau)$".

Proof of Claim. For (2) $\Rightarrow$ (1), fix p and let M be the transitive collapse of an *elementary* submodel of a sufficiently large $\mathcal{H}_\kappa$, containing all necessary parameters. Then by assumption $M \models p \Vdash \theta(\tau)$, but by elementarity and definability of $\Vdash$ in $\mathcal{H}_\kappa$, this implies $p \Vdash \theta(\tau)$ (note that this direction is trivial and does not require θ to have any particular complexity).

For (1) $\Rightarrow$ (2), assume that $p \Vdash \theta(\tau)$ and, towards contradiction, let M be such that $M \models p \nVdash \theta(\tau)$ (note that by absoluteness, $M \models p \in \mathbb{P}$ and $M \models$ "τ is a countable name for a real"). Then there is $p' \leq p$ in M such that $M \models p' \Vdash \neg\theta(\tau)$. Let $q \leq p'$ be an $(M, \mathbb{P})$-generic condition, and let G be $\mathbb{P}$-generic over V with $q \in G$. Then G is also M-generic, and $p' \in G$, hence $M[G] \models \neg\theta(\tau[G])$. But this is a $\mathbf{\Sigma}^1_2$ formula, so by upwards absoluteness $V[G] \models \neg\theta(\tau[G])$. This contradicts the assumption that $p \Vdash \theta(\tau)$. □(Claim)

The rest follows by induction.

- For $n \geq 2$, assume inductively that for $\mathbf{\Pi}^1_n$ formulas χ, the relation "$p \Vdash \chi(\tau)$" is $\mathbf{\Pi}^1_n$. Let θ be $\mathbf{\Sigma}^1_n$. Then $p \Vdash \theta(\tau)$ iff $\forall q \, (q \in \mathbb{P} \wedge q \leq p \;\rightarrow\; q \nVdash \neg\theta(\tau))$, which is easily seen to be $\mathbf{\Pi}^1_{n+1}$.
- For $n \geq 2$, assume inductively that for $\mathbf{\Sigma}^1_n$ formulas χ, the relation "$p \Vdash \chi(\tau)$" is $\mathbf{\Pi}^1_{n+1}$. Let θ be $\mathbf{\Pi}^1_{n+1}$, and write $\theta(\tau)$ as $\forall y \, \chi(\tau, y)$ for a $\mathbf{\Sigma}^1_n$ formula χ. Then the following are equivalent:

  (1) $p \Vdash \theta(\tau)$, and

  (2) $\forall q \forall \sigma \, ((q \in \mathbb{P}$ and $q \leq p$ and "σ is a countable name for a real"$) \;\rightarrow\; q \Vdash \chi(\tau, \sigma))$.

  (1) $\rightarrow$ (2) is obvious, and for (2) $\rightarrow$ (1), note that if $p \nVdash \theta(\tau)$ then $\exists q \leq p$ such that $q \Vdash \neg\theta(\tau)$, so $q \Vdash \exists y \neg\chi(\tau, y)$. But then there is a countable name σ and $q' \leq q$ such that $q' \Vdash \neg\chi(\tau, \sigma)$, which contradicts (2).

  As "$q \Vdash \chi(\tau, \sigma)$" is $\mathbf{\Pi}^1_{n+1}$ by induction, the statement in (2) is also $\mathbf{\Pi}^1_{n+1}$.

□

3.2 Iterations

Next, we consider iterations of Suslin and Suslin$^+$ proper forcing notions. This is somewhat tricky, since, in general, even a two-step iteration of Suslin$^+$ forcing notions is not Suslin$^+$ (see [34, Remark 4.12]), so the definition of the iteration cannot be absolute between countable models M and V. However, following [22] and [18], adequate preservation results can still be proved, and that is sufficient for our purposes. In this paper we will only consider iterations of length at most ω_1 with countable support. Most of our technical results just involve proper initial segments of the ω_1-iteration, which simplifies many things.

Definition 3.8. Let $\mathbb{P}_\gamma := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma \rangle$ be a countable (i.e., full) support iteration of length $\gamma < \omega_1$. We call this a *Suslin* (*Suslin*$^+$) *proper iteration* of length γ if each iterand is Suslin (Suslin$^+$) proper, i.e., for every $\alpha < \gamma$, $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathbb{Q}}_\alpha$ is Suslin (Suslin$^+$) proper".

Since the iteration $\mathbb{P}_\gamma$ is uniquely determined by the sequence $\langle \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma \rangle$ of iterands, any candidate M containing the names for the defining parameters of all the $\dot{\mathbb{Q}}_\alpha$'s can uniquely reconstruct the iteration (see e.g. [18, p. 350ff] for details). We will refer to this M-reconstruction of the iteration as $\mathbb{P}^M_\gamma$. In general, $\mathbb{P}^M_\gamma$ is not the same as $\mathbb{P}_\gamma \cap M$. Later we will prove that being a $\mathbb{P}_\gamma$-condition is $\mathbf{\Pi}^1_2$, so by downward absoluteness $\mathbb{P}^M_\gamma \supseteq \mathbb{P}_\gamma \cap M$ does hold. However, $\mathbb{P}^M_\gamma$ might contain objects which M believes to be $\mathbb{P}_\gamma$-conditions but which actually (in V) are not.

Definition 3.9 (Judah-Shelah; Goldstern; Kellner)**.** If G is a $\mathbb{P}_\gamma$-generic filter over V, and M is a candidate, we can define $G^M = G^M_\gamma$, the "potential $\mathbb{P}^M_\gamma$-generic filter over M induced by G", by induction on $\alpha \leq \gamma$, following [18, Definition 2.6] (see also [34, Definition 4.3.]).

- If $\alpha = \beta + 1$ then $G^M_\alpha = \{p \in \mathbb{P}^M_\alpha \mid p{\restriction}\beta \in G^M_\beta \text{ and } p(\beta)[G^M_\beta] \in G(\beta)\}$.
- If α is limit then $G^M_\alpha = \{p \in \mathbb{P}^M_\alpha \mid \forall \beta < \alpha \, (p \in G^M_\beta)\}$.

(here $G(\beta)$ is the β-th component of G). Then $G^M := G^M_\gamma$ is the result of this induction.

Remark 3.10.

1. The object G^M is not always well-defined: for example, if at some stage $\alpha < \gamma$, G^M_α is not $\mathbb{P}^M_\alpha$-generic over M then it does not make sense to evaluate $p(\alpha)[G^M_\alpha]$, so we cannot define $G^M_{\alpha+1}$ either. Therefore, we allow the possibility that G^M is undefined; but when we say "G^M is $\mathbb{P}^M_\gamma$-generic over M", we mean that, inductively, every G^M_α is $\mathbb{P}^M_\alpha$-generic over M for $\alpha < \gamma$, and hence every G^M_α is properly defined (and G^M is $\mathbb{P}^M_\gamma$-generic over M).
2. If G^M is well-defined then, as a filter on $\mathbb{P}^M_\gamma$, it takes the role that "$G \cap M$" would in the usual situation (i.e., where M is a collapse of an elementary submodel). In general, G^M and $G \cap M$ are different. However, their difference arises only from the difference between $\mathbb{P}^M_\gamma$ and $\mathbb{P}_\gamma \cap M$. In particular, if $p \in G^M$ and p is really a $\mathbb{P}_\gamma$-condition, then in fact $p \in G$. This follows inductively from the definition of G^M (if all initial segments of p are real $\mathbb{P}_\alpha$-conditions, then, inductively, it follows that the definition of G^M_α corresponds to the standard definition of the iterated generic filter G_α). We will need this fact several times in our arguments.

See [34] and [41] for a more detailed treatment of these issues.

Definition 3.11. Following [18, Remark 2.13] and [34, Definition 4.4], we define:

- A condition $q \in \mathbb{P}_\gamma$ is $(M, \mathbb{P}_\gamma)$-*generic* if $q \Vdash$ "$\dot{G}^M$ is a $\mathbb{P}_\gamma^M$-generic filter over M".
- If $p \in \mathbb{P}_\gamma^M$, then q is $(M, \mathbb{P}_\gamma, p)$-*generic* if it is $(M, \mathbb{P}_\gamma)$-*generic* and, additionally, $q \Vdash p \in \dot{G}^M$.

The purpose of the "$(M, \mathbb{P}_\gamma, p)$-generic condition" is that we would like to say "for $p \in \mathbb{P}_\gamma^M$, there is $q \leq p$ which is $(M, \mathbb{P}_\gamma)$-generic", but we cannot say this since p might not be in $\mathbb{P}_\gamma$. Instead, saying that "q is $(M, \mathbb{P}_\gamma, p)$-generic" is the desired analogue.

The following theorem, proved by Judah-Shelah and by Goldstern, shows that a property that is almost "properness-for-candidates" is preserved by countable support iterations of Suslin and Suslin$^+$ forcings.

Theorem 3.12 (Judah-Shelah; Goldstern). *Let $\mathbb{P}_\gamma := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma \rangle$ be a Suslin$^+$ proper iteration of length $\gamma < \omega_1$. Then for every candidate M containing the parameters of all $\dot{\mathbb{Q}}_\alpha$ and containing γ, and for every $p \in \mathbb{P}_\gamma^M$, there exists a q which is $(M, \mathbb{P}_\gamma, p)$-generic.*

Proof. This is a specific instance of a more general preservation theorem, where the iteration can have length $\gamma \leq \omega_2$. In that case, we must first make sense of the way a countable model M reflects the iteration. This general result is proved in detail for Suslin proper forcings in [22, Lemma 2.8] and in [18, Theorem 2.16, Corollary 2.17], and in [19] it is also mentioned that analogous results hold for Suslin$^+$. Even stronger results are proved by Shelah in [41], and also by Kellner in [34, Lemma 4.8]. $\square$

Remark 3.13. As we are only dealing with countable iterations, the following holds for $\mathbb{P}_\alpha$ by induction on $\alpha < \omega_1$:

1. Since by Theorem 3.12, each $\mathbb{P}_\alpha$ is proper, $\mathbb{P}_\alpha$-names for reals have countable names (modulo strengthening of the condition).
2. It follows that, inductively, we can assume that all components of $p \in \mathbb{P}_\alpha$ are represented by countable names for reals.
3. As countable names are coded by reals and α is countable, an entire condition $p \in \mathbb{P}_\alpha$ can be coded by a single real. As before, we will identify $\mathbb{P}_\alpha$-conditions and countable $\mathbb{P}_\alpha$-names for reals with the reals coding them.

Now that we can treat $\mathbb{P}_\alpha$ as a forcing with real number conditions, we can also analyze the complexity of $\mathbb{P}_\alpha$, $\leq_\alpha$ and the forcing relation $\Vdash_\alpha$. We already mentioned that $\mathbb{P}_\alpha$ is not Suslin or Suslin$^+$, i.e., neither $\mathbb{P}_\alpha$ nor $\leq_\alpha$ are $\boldsymbol{\Sigma}^1_1$. However, we can prove the following result, inductively on $\alpha < \omega_1$.

Lemma 3.14. *Let $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ be a Suslin$^+$ proper iteration. Then the following holds for all $\alpha < \omega_1$:*

1. *$\mathbb{P}_\alpha$ is $\mathbf{\Pi}^1_2$,*
2. *$\leq_\alpha$ is $\mathbf{\Pi}^1_2$,*
3. *being a countable $\mathbb{P}_\alpha$-name for a real is $\mathbf{\Pi}^1_2$, and*
4. *for any $p \in \mathbb{P}_\alpha$ and a countable $\mathbb{P}_\alpha$-name for a real τ, we have for all $n \geq 2$:*
   (a) *If θ is $\mathbf{\Pi}^1_n$ then "$p \Vdash_\alpha \theta(\tau)$" is $\mathbf{\Pi}^1_n$.*
   (b) *If θ is $\mathbf{\Sigma}^1_n$ then "$p \Vdash_\alpha \theta(\tau)$" is $\mathbf{\Pi}^1_{n+1}$.*

Proof. The case $\alpha = 1$ follows from the definition of Suslin$^+$ properness and Lemma 3.7. Assume (1)–(4) holds for $\beta < \alpha$. Then:

1. If $\alpha = \beta + 1$ then $p \in \mathbb{P}_{\beta+1}$ iff $p{\restriction}\beta \in \mathbb{P}_\beta$ and "$p(\beta)$ is a countable $\mathbb{P}_\beta$-name for a real" and $(p{\restriction}\beta) \Vdash_\beta p(\beta) \in \dot{\mathbb{Q}}_\beta$. By induction, this is a conjunction of three $\mathbf{\Pi}^1_2$ sentences, where the last one is so due to point (4) and the fact that "$p(\beta) \in \dot{\mathbb{Q}}_\beta$" is $\mathbf{\Sigma}^1_1$. If α is limit then (since we are dealing with countable support iterations and $\alpha < \omega_1$) $p \in \mathbb{P}_\alpha$ iff $\forall \beta < \alpha \ (p{\restriction}\beta \in \mathbb{P}_\beta)$. Again, this statement is $\mathbf{\Pi}^1_2$ by the induction hypothesis.

2. If $\alpha = \beta + 1$ then $p \leq_{\beta+1} q$ iff $(p{\restriction}\beta) \leq_\beta (q{\restriction}\beta)$ and $(p{\restriction}\beta) \Vdash_\beta p(\beta) \leq_{\dot{\mathbb{Q}}_\beta} q(\beta)$, which is again a conjunction of $\mathbf{\Pi}^1_2$ formulas, by induction. If α is limit then $p \leq_\alpha q$ iff $\forall \beta < \alpha \ (p{\restriction}\beta) \leq_\beta (q{\restriction}\beta)$ which is likewise $\mathbf{\Pi}^1_2$.

3. The complexity of the set of countable $\mathbb{P}_\alpha$-names is the same as the complexity of $\mathbb{P}_\alpha$, so this follows from point (1).

4. Similarly to the proof of Lemma 3.7, we prove this by induction on the complexity of θ, starting with $\mathbf{\Pi}^1_2$. As before:

   Claim. The following are equivalent:

   (a) $p \Vdash_\alpha \theta(\tau)$,

   (b) for all candidates M containing τ, p and α, and any parameters appearing in the definition of any $\dot{\mathbb{Q}}_\beta$ for $\beta < \alpha$ or in θ, we have $M \models p \Vdash_\alpha \theta(\tau)$.

   The proof of this equivalence is as in Lemma 3.7, using the "almost-properness-for-candidates"-property satisfied by $\mathbb{P}_\alpha$ (i.e., Theorem 3.12). However, since $\mathbb{P}_\alpha$ is not absolute between V and M, the argument must proceed with some more care. Notice that by downward $\mathbf{\Pi}^1_2$-absoluteness, we now already know that $\mathbb{P}_\alpha \cap M \subseteq \mathbb{P}^M_\alpha$.

   The $(b) \Rightarrow (a)$ direction is exactly as before, i.e., we simply take M to be the collapse of an elementary submodel of $\mathcal{H}_\kappa$. For $(a) \Rightarrow (b)$, assume

$p \Vdash_\alpha \theta(\tau)$ and let M be a candidate containing the relevant parameters, and, towards contradiction, suppose $M \models p \not\Vdash_\alpha \theta(\tau)$. By downward $\mathbf{\Pi}^1_2$-absoluteness we know that $p \in \mathbb{P}_\alpha^M$, and also M knows that τ is a countable name for a real.

Then $M \models \exists p' \leq_\alpha p\ (p' \Vdash_\alpha \neg\theta(\tau))$ (note that p' may not be in $\mathbb{P}_\alpha$). Now use Theorem 3.12, and find a condition $q \in \mathbb{P}_\alpha$ which is $(M, \mathbb{P}_\alpha, p)$-generic. Then, if G is $\mathbb{P}_\alpha$-generic over V, and $q \in G$, the derived object G^M (see Definition 3.9 (1)) is $\mathbb{P}_\alpha^M$-generic over M, and $p' \in G^M$. Therefore, $M[G^M] \models \neg\theta(\tau[G])$, and by upward $\mathbf{\Sigma}^1_2$-absoluteness, $\neg\theta(\tau[G])$ holds in $V[G]$. But also $M \models p' \leq_\alpha p$ and $G^M \subseteq \mathbb{P}_\alpha^M$ is a filter, so also $p \in G^M$. But p was in $\mathbb{P}_\alpha$, so by Remark 3.10 (2) p must in fact be in G. That contradicts $p \Vdash_\alpha \theta(\tau)$. □(Claim).

For the rest, proceed inductively as before: if θ is $\mathbf{\Sigma}^1_n$ for $n \geq 2$ then $p \Vdash_\alpha \theta(\tau)$ iff $\forall q\ (q \in \mathbb{P}_\alpha$ and $q \leq_\alpha p \ \rightarrow\ q \not\Vdash_\alpha \neg\theta(\tau))$, which is $\mathbf{\Pi}^1_{n+1}$, using the fact that $\mathbb{P}_\alpha$ and $\leq_\alpha$ are $\mathbf{\Pi}^1_2$, i.e., points (1) and (2) of the theorem. Likewise, if $\theta(\tau) \equiv \forall y\, \chi(\tau, y)$ is $\mathbf{\Pi}^1_{n+1}$ for $n \geq 2$ and χ is $\mathbf{\Sigma}^1_n$, then, as before, $p \Vdash_\alpha \theta(\tau)$ iff $\forall q \forall \sigma\ ((q \in \mathbb{P}_\alpha$ and $q \leq_\alpha p$ and "σ is a countable $\mathbb{P}_\alpha$-name for a real") $\rightarrow\ q \Vdash_\alpha \chi(\tau, \sigma))$. Again this is $\mathbf{\Pi}^1_{n+1}$ using the fact that being a countable $\mathbb{P}_\alpha$-name for a real is $\mathbf{\Pi}^1_2$, i.e., point (3) of the theorem. □

From this theorem it follows that $\mathbb{P}_\alpha$, $\leq_\alpha$, being a countable $\mathbb{P}_\alpha$-name for a real, and the relation "$p \Vdash_\alpha \theta(\tau)$" for $\mathbf{\Pi}^1_2$ formulas θ, are all downwards absolute between V and countable models M (containing the relevant parameters), and absolute in both directions between V and models $W \subseteq V$ with $\omega_1 \subseteq W$.

3.3 Suslin$^+$ proper iterations and inaccessibles.

We end this section with two further useful results about Suslin$^+$ proper forcing, under the assumption that $\forall r\ (\omega_1^{L[r]} < \omega_1)$. The first result shows that this assumption is preserved by Suslin$^+$ iterations of countable length.

First, a preliminary Lemma.

Lemma 3.15. *Suppose $V \models \forall r\ (\omega_1^{L[r]} < \omega_1)$, $\mathbb{P}_\gamma := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma \rangle$ is a Suslin$^+$ proper iteration of length $\gamma < \omega_1$, and p is a $\mathbb{P}_\gamma$-condition. Suppose further that p and all the defining parameters of $\mathbb{P}_\gamma$ are coded by a real z. Then there exists $q \in \mathbb{P}_\gamma$ such that $q \Vdash_\gamma p \in \dot{G}$ and $q \Vdash_\gamma$ "$\dot{G}$ is $L[\check{z}]$-generic".*

Proof. As ω_1^V is inaccessible in $L[z]$, we can find a candidate $M \subseteq L[z]$ containing all the reals and dense sets of $L[z]$, and moreover reflecting all the relevant properties of $L[z]$ (for example, let $M := L_\lambda[z]$ for some sufficiently large $\lambda < \omega_1^V$ such that $L_\lambda[z] \prec L_{\omega_1^V}[z]$). As p and the parameters of $\mathbb{P}_\gamma$ are now in M, by Theorem 3.12 we can find an $(M, \mathbb{P}_\gamma, p)$-generic condition q. Then $q \Vdash_\gamma$ "$\dot{G}^M$ is $\mathbb{P}_\gamma^M$-generic over M" and, since M has the same reals and dense sets as $L[z]$, also $q \Vdash_\gamma$ "$\dot{G}$ is $L[\check{z}]$-generic". On the other hand, $q \Vdash_\gamma p \in \dot{G}^M$, and since $p \in \mathbb{P}_\gamma$, this implies $q \Vdash_\gamma p \in \dot{G}$ by Remark 3.10 (2). $\square$

Theorem 3.16. *Suppose $V \models \forall r\, (\omega_1^{L[r]} < \omega_1)$ and $\mathbb{P}_\gamma := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \gamma \rangle$ is a Suslin$^+$ proper iteration of length $\gamma < \omega_1$. Then $V^{\mathbb{P}_\gamma} \models \forall r\, (\omega_1^{L[r]} < \omega_1)$.*

Proof. Suppose, towards contradiction, that the conclusion is false, and let $\dot{r}$ be a countable $\mathbb{P}_\gamma$-name for a real and $p \in \mathbb{P}_\gamma$ such that $p \Vdash_\gamma \omega_1^{L[\dot{r}]} = \omega_1$. Let z be a real in V, coding $p, \dot{r}$ and all the defining parameters of $\mathbb{P}_\gamma$. By Lemma 3.15, there is a $q \in \mathbb{P}_\gamma$ such that $q \Vdash_\gamma p \in \dot{G}$ and $q \Vdash_\gamma$ "$\dot{G}$ is $L[\check{z}]$-generic".

By Remark 3.5, we know that every iterand occurring in $\mathbb{P}_\gamma$ is Suslin$^+$ proper in $L[z]$ as well, so $L[z] \models$ "$\mathbb{P}_\gamma$ is proper". Therefore $q \Vdash_\gamma$ "$L[\check{z}][\dot{G}]$ is a proper forcing extension of $L[\check{z}]$", so in particular

$$q \Vdash_\gamma \omega_1^{L[\check{z}][\dot{G}]} = \omega_1^{L[\check{z}]}.$$

Now notice that in any $\mathbb{P}_\gamma$-extension $V[G]$ of V, since $\dot{r}[G]$ is constructible from $\dot{r}$ and G, and $\dot{r}$ is coded in z, we know that $L[\dot{r}[G]] \subseteq L[z][G]$, which implies $\omega_1^{L[\dot{r}[G]]} \leq \omega_1^{L[z][G]}$. On the other hand, $\omega_1^{L[z]}$ was countable by assumption. It follows that

$$q \Vdash_\gamma \omega_1^{L[\dot{r}]} \leq \omega_1^{L[\check{z}][\dot{G}]} = \omega_1^{L[\check{z}]} < \omega_1$$

which, together with $q \Vdash_\gamma p \in \dot{G}$ and $p \Vdash_\gamma \omega_1^{L[\dot{r}]} = \omega_1$, leads to a contradiction. $\square$

Remark 3.17. The definability of the forcing is essential in the preceding result, since, in general, the assumption $\forall r\, (\omega_1^{L[r]} < \omega_1)$ is not preserved even by ccc forcings. For example, assuming that ω_1 is not Mahlo in L, one can find $A \subseteq \omega_1$ is such that $L[A] \models \forall r (\omega_1^{L[r]} < \omega_1)$. Then, using the technique of *almost disjoint coding* (see e.g. [25]), one can construct a ccc forcing notion coding A by the generic real r_G. So $L[A][r_G] = L[r_G]$ will be a generic extension of $L[A]$ by a ccc forcing notion, while obviously satisfying $\omega_1^{L[r_G]} = \omega_1$.

Our second result (which uses the previous result) is a strong absoluteness property of extensions by Suslin$^+$ iterations.

Definition 3.18. Let $\mathbb{P}$ be a forcing notion. Then

1. V is $\mathbf{\Sigma}^1_n$-$\mathbb{P}$-*absolute* iff for all $\mathbf{\Sigma}^1_n$ formulas ϕ, all $\mathbb{P}$-generic G over V, and all reals $x \in V$:
$$V \models \phi(x) \iff V[G] \models \phi(x).$$
2. V is $\mathbf{\Sigma}^1_n$-$\mathbb{P}$-*correct* iff for all $\mathbf{\Sigma}^1_n$ formulas ϕ, all $\mathbb{P}$-generic G over V, and all reals $x \in V[G]$:
$$V[x] \models \phi(x) \iff V[G] \models \phi(x).$$

$\mathbf{\Sigma}^1_n$-$\mathbb{P}$-correctness implies $\mathbf{\Sigma}^1_n$-$\mathbb{P}$-absoluteness, but not vice versa. In fact, $\mathbf{\Sigma}^1_n$-$\mathbb{P}$-correctness is much stronger. The following is clear:

Fact 3.19. *If V is $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-correct, then $\mathbf{\Sigma}^1_3$-absoluteness holds between any two models W and W' with $V \subseteq W \subseteq W' \subseteq V[G]$.*

Proof. Let ϕ be $\mathbf{\Sigma}^1_3$ and $x \in W$. If $W \models \phi(x)$ then $W' \models \phi(x)$ by upwards $\mathbf{\Sigma}^1_3$-absoluteness (i.e., Shoenfield absoluteness). Conversely, if $W' \models \phi(x)$ then by upwards $\mathbf{\Sigma}^1_3$-absoluteness $V[G] \models \phi(x)$, so by $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-correctness $V[x] \models \phi(x)$, so by upwards-absoluteness again $W \models \phi(x)$. □

$\mathbf{\Sigma}^1_n$-$\mathbb{P}$-correctness and $\mathbf{\Sigma}^1_n$-$\mathbb{P}$-absoluteness for all set-forcings $\mathbb{P}$ have been investigated before, by Woodin, Bagaria and Friedman among others. For instance, in [1] and [13] it is shown that $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness for all set-forcings $\mathbb{P}$ can be obtained from a reflecting cardinal, whereas $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-correctness for all set-forcings $\mathbb{P}$ implies the existence of sharps for sets of ordinals by [46]).

If we restrict attention to Suslin$^+$ proper forcing notions, $\mathbf{\Sigma}^1_3$-correctness can be obtained just from an inaccessible. In fact, in [3, Lemma 9.5.4] it is proved that if $V \models \forall r\, (\omega_1^{L[r]} < \omega_1)$ and $\mathbb{P}$ is Suslin ccc, then V is $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-correct. We now extend this result to all Suslin$^+$ proper forcings $\mathbb{P}$ (and their iterations of length ω_1), relying on Theorem 3.16.

Theorem 3.20. *Suppose $V \models \forall r\, (\omega_1^{L[r]} < \omega_1)$ and $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ is a Suslin$^+$ proper iteration. Then V is $\mathbf{\Sigma}^1_3$-$\mathbb{P}_{\omega_1}$-correct.*

Proof. Since upwards $\mathbf{\Sigma}^1_3$-absoluteness always holds, it remains to prove the converse. First we do it for countable iterations $\mathbb{P}_\gamma$, $\gamma < \omega_1$.

Suppose, towards contradiction, that downwards-$\mathbf{\Sigma}^1_3$-$\mathbb{P}_\gamma$-correctness fails. Then there is a $\mathbf{\Sigma}^1_3$ formula ϕ, a countable $\mathbb{P}_\gamma$-name for a real τ, and a condition $p \in \mathbb{P}_\gamma$ such that

$$(*) \qquad\qquad p \Vdash_\gamma (\phi(\tau) \;\wedge\; V[\tau] \models \neg\phi(\tau)).$$

Our goal is to contradict $(*)$. Let θ be a $\mathbf{\Pi}^1_2$ formula and σ a (without loss of generality countable) $\mathbb{P}_\gamma$-name for a real, such that

$$p \Vdash_\gamma (\theta(\tau, \sigma) \;\wedge\; V[\tau] \models \neg\phi(\tau)).$$

Let z be a real coding τ, σ, p and all the defining parameters appearing in $\mathbb{P}_\gamma$ and in ϕ. By Lemma 3.15, there is a q forcing "$\dot{G}$ is $L[\check{z}]$-generic" and "$p \in \dot{G}$". Let G_γ be any such generic filter with $q \in G_\gamma$ and let us work in $V[G_\gamma]$ for the time being.

Let $x := \tau[G_\gamma]$ and $y = \sigma[G_\gamma]$. Since $p \in G_\gamma$, by $(*)$ we know that $V[G_\gamma] \models \theta(x, y)$. By Shoenfield absoluteness, we also know that $L[z][x][y] \models \theta(x, y)$. As $L[z][G_\gamma]$ is a generic extension of $L[z]$, we know that the intermediary models $L[z] \subseteq L[z][x] \subseteq L[z][x][y] \subseteq L[z][G_\gamma]$ can all be represented by generic extensions. Let $\mathbb{Q}$ be the forcing leading from $L[z][x]$ to $L[z][x][y]$ (to find $\mathbb{Q}$, first look at the quotient of $\mathbb{P}_\gamma$ modulo the sub-forcing generated by τ, and then take the sub-forcing of that generated by σ). It follows that

$$L[z][x] \models \exists q \in \mathbb{Q}\,(q \Vdash_{\mathbb{Q}} \theta(\check{x}, \sigma)).$$

But by Theorem 3.16, $V[G_\gamma] \models \forall r\,(\omega_1^{L[r]} < \omega_1)$. Therefore also $V[x] \models \forall r\,(\omega_1^{L[r]} < \omega_1)$. Therefore, in particular, $V[x] \models$ "ω_1 is inaccessible in $L[z][x]$". So, in $V[x]$, we can find an *internal* $\mathbb{Q}$-generic filter H over $L[z][x]$, so $V[x] \models (L[z][x][H] \models \theta(x, \sigma[H]))$. By upwards-absoluteness, $V[x] \models \exists y' \theta(x, y')$, i.e., $V[x] \models \phi(x)$. But this is a contradiction with $(*)$, since we had $p \Vdash V[\tau] \models \neg\phi(\tau)$.

To complete the proof of the theorem, it only remains to verify $\mathbf{\Sigma}^1_3$-correctness for the entire iteration of length ω_1. But obviously, if $V[G_{\omega_1}] \models \phi(x)$ for some $\mathbf{\Sigma}^1_3$ formula ϕ, then actually $V[G_{\omega_1}] \models \exists y \theta(x, y)$, and since $\mathbb{P}_{\omega_1}$ is proper, x and y must both appear at some stage $\gamma < \omega_1$, so by Shoenfield absoluteness $V[G_\gamma] \models \theta(x, y)$. Then, by what we have proved above, $V[x] \models \phi(x)$. □

4 Methods for obtaining regularity.

The purpose of this section is to develop methods for obtaining regularity for $\mathbf{\Delta}^1_3$ sets of reals, but doing this with "as little damage as possible", i.e., using forcing iterations that preserve certain properties of the ground model. In total, we will present three separate methods of achieving this goal. The first one is due to Judah:

Theorem 4.1 (Judah)**.**

1. *If $V \models \mathbf{\Sigma}^1_2(\mathbb{B})$ and $\mathbb{B}_{\omega_1}$ denotes the ω_1-product of random forcing, then $V^{\mathbb{B}_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{B})$.*

2. *If $V \models \mathbf{\Sigma}^1_2(\mathbb{C})$ and $\mathbb{C}_{\omega_1}$ denotes the ω_1-product of Cohen forcing, then $V^{\mathbb{C}_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{C})$.*

Proof. See [3, Theorem 9.4.6]. □

It is not clear whether the above can be generalized beyond Cohen and random: the proofs depend on properties of the meager and null ideals (such as the Fubini property), as well as a strong homogeneity of Cohen- and random-products.

Next, we present our second method for obtaining $\mathbf{\Delta}^1_3$-regularity. This is inspired by Shelah's original proof that $\mathbf{\Delta}^1_3(\mathbb{B})$ does not require an inaccessible, see [40, §6].

Although we are primarily interested in the regularity properties mentioned in Section 2, we would like our proofs to be sufficiently uniform and general, i.e., we would like them to be applicable to many forcing notions $\mathbb{P}$ at once. We could require that $\mathbb{P}$ has trees as conditions (see e.g. *arboreal forcing* from

[6, 24]), but in some cases (e.g. Cohen) we prefer to work with G_δ sets instead. The reason is that, otherwise, we would need to work "modulo an ideal" which would only complicate the proofs unnecessarily. So we relax the requirement somewhat and adopt the following:

Convention and Notation 4.2.

- Let us say that $\mathbb{P}$ is a *real forcing notion* if the conditions are G_δ sets of reals, ordered by inclusion. For conditions $p \in \mathbb{P}$, we will generally use "p" to refer to the real number coding the condition (i.e., G_δ code), and "$[p]$" to refer to the corresponding set of reals. When $\mathbb{P}$-conditions are closed sets, we may identify p with a tree and $[p]$ with the set of branches through that tree.

- We assume that, as usual, $\mathbb{P}$ adds a generic real, denoted by $\dot{g}$, and that for all $p \in \mathbb{P}$ we have $\Vdash_{\mathbb{P}} (p \in \dot{G} \leftrightarrow \dot{g} \in [p])$ (so the generic filter and real are mutually reconstructible).

- We will also assume that for projective pointclasses $\mathbf{\Gamma}$, the statements
  - "$\forall A \in \mathbf{\Gamma}\ \forall p \in \mathbb{P}\ \exists q \leq p\ ([q] \subseteq A$ or $[q] \cap A = \varnothing)$" and
  - "$\forall A \in \mathbf{\Gamma}\ \exists p\ ([p] \subseteq A$ or $[p] \cap A = \varnothing)$"

  are equivalent, so that, as in Observation 2.7, in order to prove $\mathbf{\Gamma}(\mathbb{P})$ it will suffice to prove the latter statement (this is achieved by using suitable homeomorphisms between $[p]$ for conditions $p \in \mathbb{P}$ and the entire space of reals).

This level of generality will certainly take care of everything we are interested in, and potentially much more. Note that we could be even more lenient in the convention and allow the conditions of $\mathbb{P}$ to be Borel sets that are large with respect to an ideal, following the approach of Zapletal [47].

Before stating the next theorem we introduce *amoebas* and *quasi-amoebas* for real forcing notions.

Definition 4.3. Let $\mathbb{P}$ be a real forcing notion, and $\mathbb{Q}$ another forcing. We say that

1. $\mathbb{Q}$ is a *quasi-amoeba for* $\mathbb{P}$ if for every $p \in \mathbb{P}$ and every $\mathbb{Q}$-generic G, there is a $q \in \mathbb{P}^{V[G]}$ such that $q \leq_{\mathbb{P}} p$ and
$$V[G] \models \forall x \in [q]\ (x \text{ is } \mathbb{P}\text{-generic over } V).$$

2. $\mathbb{Q}$ is an *amoeba for* $\mathbb{P}$ if for every $p \in \mathbb{P}$ and every $\mathbb{Q}$-generic G, there is a $q \in \mathbb{P}^{V[G]}$ such that $q \leq_{\mathbb{P}} p$ and for any larger model $W \supseteq V[G]$,
$$W \models \forall x \in [q]\ (x \text{ is } \mathbb{P}\text{-generic over } V).$$

There is a subtle difference between amoebas and quasi-amoebas, which is not visible in the Cohen and random (and, in general, ccc) case, because the assertion "$[q]$ consists of Cohen/random reals over V" is upwards absolute for Cohen/random-conditions q. For non-ccc forcing this is not always the case: for example, "T is a perfect tree of Sacks reals over V" is not upwards absolute, as shown in the next example.

Example 4.4.

1. $\mathbb{A}$ (the standard *amoeba for measure*) is an amoeba for $\mathbb{B}$ (see [3, Section 3.4]).
2. $\mathbb{UM}$ (the standard *amoeba for category*) is an amoeba for $\mathbb{C}$ (see [40, §4]). Also, if $\mathbb{D}$ is Hechler forcing, then the two-step iteration $(\mathbb{D} * \mathbb{D})$ is an amoebas for $\mathbb{C}$ (see [3, Theorem 3.5.1]).
3. Mathias forcing $\mathbb{R}$ is an amoeba for itself ([38, Corollary 2.5]).
4. $\mathbb{S}$ is a quasi-amoeba, but not an amoeba, for itself ([4, Theorem 4, Corollary 5]).
5. $\mathbb{M}$ is a quasi-amoeba, but not an amoeba, for itself ([4, Proposition 7]).
6. $\mathbb{L}$ is *not* a quasi-amoeba for itself ([4, Theorem 5]).

One might expect quasi-amoebas to be quite useless in iterated forcing constructions, since the property of adding large sets of generic reals is only temporary. Nevertheless, the success of our methods is in part due to the realization that quasi-amoebas are, in fact, sufficient for the following argument.

Theorem 4.5. *Suppose $\mathbb{P}$ is a real forcing notion and $\mathbb{AP}$ a quasi-amoeba for $\mathbb{P}$. Furthermore, assume that both $\mathbb{P}$ and $\mathbb{AP}$ are Suslin$^+$ proper. Let $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ be a countable support iteration whose iterands are $\mathbb{P}$ and $\mathbb{AP}$ interlaced (i.e., for even α, $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha \cong \mathbb{P}$ and for odd α, $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha \cong \mathbb{AP}$). Then $V^{\mathbb{P}_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{P})$.*

Proof. Let G_{ω_1} be $\mathbb{P}_{\omega_1}$-generic over V, let $A = \{x \mid \phi(x)\} = \{x \mid \neg\psi(x)\}$ be a $\mathbf{\Delta}^1_3$ set in $V[G_{\omega_1}]$, defined by $\mathbf{\Sigma}^1_3$-formulas ϕ and ψ. As our iteration is proper, we may assume, without loss of generality, that the parameters appearing in ϕ and ψ are in the ground model V (otherwise, they are in some $V[G_{\alpha_0}]$, and we repeat the same argument with $V[G_{\alpha_0}]$ as the ground model).

Our goal is to find a $\mathbb{P}$-condition p in $V[G_{\omega_1}]$ such that $[p] \subseteq A$ or $[p] \cap A = \varnothing$.

Let x_0 be the $\mathbb{P}$-generic real over V, added at the first step of the iteration. In $V[G_{\omega_1}]$, either $\phi(x_0)$ or $\psi(x_0)$ must hold, so without loss of generality we assume that $\phi(x_0)$ holds. Then $\exists y \theta(x_0, y)$ holds for some $\mathbf{\Pi}^1_2$ formula θ such that $\phi(x_0) \equiv \exists y \theta(x_0, y)$. By properness, there is an $\alpha < \omega_1$ such that $y \in V[G_\alpha]$, and by Shoenfield absoluteness $V[G_\alpha] \models \theta(x_0, y)$. In V, let p be a $\mathbb{P}_\alpha$-condition and τ a countable $\mathbb{P}_\alpha$-name for a real, such that

$$p \Vdash_\alpha \theta(\dot{g}_0, \tau)$$

where $\dot{g}_0$ is the name for the first $\mathbb{P}$-generic real.

Let us adopt the following notation: let $\mathbb{P}_{1,\alpha}$ be the quotient of the iteration (i.e., such that $\mathbb{P}_1 * \mathbb{P}_{1,\alpha} \cong \mathbb{P}_\alpha$), and when x is a $\mathbb{P}$-generic real over V, "$p[x]$" refers to the $\mathbb{P}_{1,\alpha}$-condition that remains of p after evaluating it according to x (i.e., the filter G_x generated by x), and "$\tau[x]$" refers to the $\mathbb{P}_{1,\alpha}$-name that remains of τ after evaluating it according to x. Here by "$\mathbb{P}_{1,\alpha}$" we are, of course, referring to the *definition* of the iteration. It is well-known that, if we consider τ and p as coded by reals (in some explicit way), then there are Borel functions mapping $\tau \mapsto \tau[x]$ and $p \mapsto p[x]$, in any model that contains x (this is similar to, e.g., [47, Proposition 2.3.1]).

Let $\tilde{\theta}(x, p, \tau)$ be a conjunction of the following statements:

- "$p[x]$ is a $\mathbb{P}_{1,\alpha}$-condition",
- "$\tau[x]$ is a countable $\mathbb{P}_{1,\alpha}$-name for a real", and
- $p[x] \Vdash_{1,\alpha} \theta(\check{x}, \tau[x])$.

Since the quotient $\mathbb{P}_{1,\alpha}$ is a Suslin$^+$ proper iteration, using Lemma 3.14 (1), (3) and (4), we conclude that $\tilde{\theta}$ is a $\mathbf{\Pi}^1_2$ statement. For convenience, we will suppress the parameters p and τ from $\tilde{\theta}$ (remember that they are in the ground model V).

As we have $p \Vdash \theta(\dot{g}_0, \tau)$ and x_0 is $\mathbb{P}$-generic over V, we have

$$V[x_0] \models \tilde{\theta}(x_0).$$

Therefore, going back to V, we have

$$p(0) \Vdash_{\mathbb{P}} \tilde{\theta}(\dot{g}_0).$$

But by Lemma 3.14 (4), the above statement is again $\mathbf{\Pi}^1_2$, so by Shoenfield absoluteness, $V[x_0] \models p(0) \Vdash_{\mathbb{P}} \tilde{\theta}(\dot{g}_0)$. Let H_1 be the next $\mathbb{AP}$-generic over $V[x_0]$ (i.e., $V[x_0][H_1] = V[G_2]$). By the definition of a quasi-amoeba, in $V[x_0][H_1]$ there is a $\mathbb{P}$-condition q, such that $q \leq p(0)$ and

$$V[x_0][H_1] \models \forall x \in [q]\ (x \text{ is } \mathbb{P}\text{-generic over } V[x_0]).$$

Then

$$V[x_0][H_1] \models \forall x \in [q]\ (V[x_0][x] \models \tilde{\theta}(x)),$$

and by $\mathbf{\Pi}^1_2$-absoluteness between $V[x_0][x]$ and $V[x_0][H_1]$:

$$V[x_0][H_1] \models \forall x \in [q]\ (\tilde{\theta}(x)).$$

Let $\Theta(q)$ abbreviate "$\forall x \in [q]\ (\tilde{\theta}(x))$", and notice that, again, it is $\mathbf{\Pi}^1_2$. This is the key step of our proof, since now, in all larger models $V[G_\beta]$, $2 \leq \beta < \omega_1$, we have

$$V[G_\beta] \models \Theta(q).$$

It remains to show that $V[G_{\omega_1}] \models [q] \subseteq A$, which will complete the proof. So, in $V[G_{\omega_1}]$, let z be any real in $[q]$. Let $\beta < \omega_1$ be such that $z \in V[G_\beta]$, and assume β is odd (so that $\beta+1$ is even). Since $V[G_{\beta+1}] \models \Theta(q)$, in particular, $V[G_{\beta+1}] \models \tilde{\theta}(z)$. But looking at the meaning of $\tilde{\theta}$, in particular it says "$p[z] \Vdash_{\mathbb{P}_{1,\alpha}} \theta(\check{z}, \tau)$", which implies "$p[z] \Vdash_{\mathbb{P}_{1,\alpha}} \exists y' \theta(\check{z}, y')$" and hence "$p[z] \Vdash_{\mathbb{P}_{1,\alpha}} \phi(\check{z})$". Notice that, by genericity, we may assume that β was chosen to be sufficiently large so that $p[z]$ in fact belongs to $G_{[\beta+1,\beta+\alpha)}$ (the generic filter restricted to stages $[\beta+1, \beta+\alpha)$ of the iteration).

It follows that $V[G_{\beta+\alpha}] \models \phi(z)$, and by upwards-absoluteness, $V[G_{\omega_1}] \models \phi(z)$. This completes the proof. □

Corollary 4.6. *If $\mathbb{P}$ is Suslin$^+$ proper and a quasi-amoeba for itself, then $V^{\mathbb{P}_{\omega_1}} \models \boldsymbol{\Delta}^1_3(\mathbb{P})$. In particular $V^{\mathbb{S}_{\omega_1}} \models \boldsymbol{\Delta}^1_3(\mathbb{S})$ and $V^{\mathbb{M}_{\omega_1}} \models \boldsymbol{\Delta}^1_3(\mathbb{M})$.*

If we want to obtain $\boldsymbol{\Delta}^1_3(\mathbb{P})$ for several different $\mathbb{P}$ at the same time, we can alter the above construction somewhat, by interlacing more forcing notions. The only requirement is that the iteration is sufficiently "repetitive", in the sense of the following definition:

Definition 4.7. Suppose $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ is a Suslin$^+$ proper iteration, where all iterands have parameters in the ground model. Such an iteration is called *repetitive* if for any $\alpha < \beta < \omega_1$, there are unboundedly many $\gamma < \omega_1$ such that

$$\langle \dot{\mathbb{Q}}_\xi \mid \alpha < \xi \leq \beta \rangle = \langle \dot{\mathbb{Q}}_{\gamma+\xi} \mid \alpha < \xi \leq \beta \rangle.$$

The following theorem is a stronger version of Theorem 4.5:

Theorem 4.8. *Suppose $\mathbb{P}$ and $\mathbb{AP}$ are as in Theorem 4.5, $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ is Suslin$^+$ proper with parameters in the ground model and repetitive, and both $\mathbb{P}$ and $\mathbb{AP}$ appear cofinally often in the iteration. Then $V^{\mathbb{P}_{\omega_1}} \models \boldsymbol{\Delta}^1_3(\mathbb{P})$.*

Proof. The proof is exactly the same as that of Theorem 4.5. Instead of looking at stages 1 and 2 of the iteration, we look at some stages α_0 and α_1. Then we find the condition q in $V[G_{\alpha_1}]$ and $\Theta(q)$ holds from that point onwards. Later we find a sufficiently large γ so that the segment $\mathbb{P}_{\alpha_1,\alpha_1+\alpha}$ is "copied" after γ, and rely on the same arguments as before. The details are left to the reader. □

In our applications, the last theorem will only be used when we have a finite number of $\mathbb{P}_i$ and quasi-amoeabs $\mathbb{AP}_i$, $i \leq k$. After iterating with $(\mathbb{P}_0 * \mathbb{AP}_0 * \cdots * \mathbb{P}_k * \mathbb{AP}_k)_{\omega_1}$ we obtain a model where $\boldsymbol{\Delta}^1_3(\mathbb{P}_i)$ holds for all $i \leq k$.

Our third method for obtaining $\boldsymbol{\Delta}^1_3$-regularity works under the assumption $\forall r\ (\omega_1^{L[r]} < \omega_1)$. Let us first mention an observation essentially due to Zapletal (cf. [47, Proposition 2.2.2.]), showing that for real forcing notions $\mathbb{P}$, when we have an $(M, \mathbb{P})$-generic condition, we can assume, without loss of generality, that *all* reals in this condition are M-generic.

Lemma 4.9 (Zapletal)**.** *Let $\mathbb{P}$ be a proper, real forcing notion, and M a countable model. If q is an $(M,\mathbb{P})$-generic condition, then there is $q' \leq q$ such that (in V) all $x \in [q]$ are M-generic.*

Proof. Let $B := \{x \in [q] \mid x \text{ is } M\text{-generic}\}$. As M is countable, it is easy to see that B is Borel. Let $\dot{g}$ be the name for the $\mathbb{P}$-generic real. Since q is $(M,\mathbb{P})$-generic, $q \Vdash \dot{g} \in B$. But Borel sets are $\mathbb{P}$-measurable (in the sense of Definition 2.1). So either there exists a $q' \leq q$ such that $[q'] \subseteq B$, in which case we are done, or, for every $q' \leq q$ there exists $q'' \leq q'$ such that $[q''] \cap B = \varnothing$. But the latter case implies that $\{q' \mid [q'] \cap B = \varnothing\}$ is dense below q, hence $q \Vdash \dot{g} \notin B$, yielding a contradiction. □

Theorem 4.10. *Suppose $V \models \forall r\, (\omega_1^{L[r]} < \omega_1)$, $\mathbb{P}$ is a real forcing notion, and $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1\rangle$ is a Suslin$^+$ proper iteration in which $\mathbb{P}$ appears cofinally often. Then $V^{\mathbb{P}_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{P})$.*

Proof. Let $A = \{x \mid \phi(x)\} = \{x \mid \neg\psi(x)\}$ be a $\mathbf{\Delta}^1_3$ set in $V[G_{\omega_1}]$. As the defining parameter appears at some initial stage of the iteration, and by Theorem 3.16 we know that $\forall r(\omega_1^{L[r]} < \omega_1)$ holds in all $V[G_\alpha]$, let us again assume, without loss of generality, that the parameters are in the ground model V. Also, without loss of generality, we may assume that the first step of the iteration is $\mathbb{P}$.

Let x_0 be the $\mathbb{P}$-generic real over V. Again, let us assume $V[G_{\omega_1}] \models \phi(x_0)$ (without loss of generality). Now by Theorem 3.20 V is $\mathbf{\Sigma}^1_3$-$\mathbb{P}_{\omega_1}$-correct, therefore $V[x_0] \models \phi(x_0)$. Then in V, there is a $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}} \phi(\dot{g})$. Then also $p \Vdash_{\mathbb{P}} \theta(\dot{g}, \tau)$ for some countable name τ and a $\mathbf{\Pi}^1_2$ formula θ such that $\phi(x) \equiv \exists y \theta(x,y)$. Now let z be a real coding p, τ, and the parameters of $\mathbb{P}$ and θ. By $\mathbf{\Pi}^1_2$-absoluteness, $L[z] \models p \Vdash_{\mathbb{P}} \theta(\dot{g}, \tau)$.

Since ω_1^V is inaccessible in $L[z]$, find a countable model $M \subseteq L[z]$ reflecting everything about $\mathbb{P}^{L[z]}$ and containing all the parameters (as in the proof of Lemma 3.15). By Lemma 4.9 there is $q \leq p$ such that all x in $[q]$ are M-generic, hence $L[z]$-generic. So (in V) for all $x \in [q]$ we have $L[z][x] \models \theta(x, \tau[x])$, and by $\mathbf{\Pi}^1_2$-absoluteness

$$V \models \forall x \in [q]\, \theta(x, \tau[x]).$$

As this statement is $\mathbf{\Pi}^1_2$, it holds in $V[G_{\omega_1}]$, so also the statement $\forall x \in [q] \exists y \theta(x,y)$ holds, so $[q] \subseteq A$. □

The advantage of this method over the one before is that we can avoid amoebas, which is useful in situations where no suitable amoebas are available, or those that are available fail to have nice properties. However, to do this we pay the price of using an inaccessible, rather than obtaining a proof on the basis of ZFC alone.

5 Completing the $\mathbf{\Delta}^1_3$-diagram.

Figure 3 shows the diagram of implications for regularity properties on the $\mathbf{\Delta}^1_3$-level.

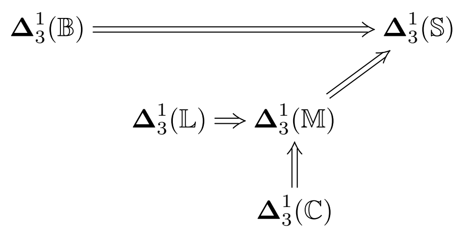

Figure 3: Diagram of implications for $\mathbf{\Delta}^1_3$ sets of reals.

We will now apply the techniques presented in the previous section to show that this diagram is complete, by constructing models, in ZFC or ZFC with an inaccessible, which separate the regularity statements. We have three methods (Theorem 4.1, Theorem 4.8 and Theorem 4.10) at our disposal for proving that $\mathbf{\Delta}^1_3(\mathbb{P})$ is true in a model. But to separate regularity properties we need another ingredient, namely, a method for showing that $\mathbf{\Delta}^1_3(\mathbb{P})$ is *false* in a given model. For this, we note that one direction in the original characterization theorems 2.3, 2.4 and 2.5 can easily be generalized (for $\mathbb{C}$ and $\mathbb{B}$ this was already mentioned and used in [2]).

Definition 5.1. A wellorder $\preceq$ of a set of reals, of length ω_1, is called $\mathbf{\Sigma}^1_n$*-good* if

1. $\preceq$ is a $\mathbf{\Sigma}^1_n$-relation, and
2. the statement "x codes the set of $\preceq$-predecessors of y" is $\mathbf{\Sigma}^1_n$.

We say that $\preceq$ is a $\mathbf{\Sigma}^1_n$*-good wellorder of the reals* if it is a wellorder of the set of all reals.

Fact 5.2. *Suppose M is a model with a $\mathbf{\Sigma}^1_n$-good wellorder of the reals. Then:*

1. $\mathbf{\Sigma}^1_n(\mathbb{B}) \implies \{x \mid x \text{ is not random over } M\} \in \mathcal{N}$.
2. $\mathbf{\Sigma}^1_n(\mathbb{C}) \implies \{x \mid x \text{ is not Cohen over } M\} \in \mathcal{M}$.
3. $\mathbf{\Delta}^1_n(\mathbb{B}) \implies \exists x\,(x \text{ is random over } M)$.
4. $\mathbf{\Delta}^1_n(\mathbb{C}) \implies \exists x\,(x \text{ is Cohen over } M)$.
5. $\mathbf{\Delta}^1_n(\mathbb{L}) \implies \exists x\,(x \text{ is dominating over } M)$.
6. $\mathbf{\Delta}^1_n(\mathbb{M}) \implies \exists x\,(x \text{ is unbounded over } M)$.
7. $\mathbf{\Delta}^1_n(\mathbb{S}) \implies \exists x\,(x \notin M)$.

Proof. Points 1–4 follow from the original proofs of Solovay and Judah-Shelah; see also [2, Lemmas 2.3, 2.85 and 2.105]. For 5 and 6, use an argument analogous to the one in [8, Theorems 4.1 and 6.1] replacing $\mathbf{\Sigma}^1_2$ by $\mathbf{\Sigma}^1_3$ and $\mathbf{\Delta}^1_2$ by $\mathbf{\Delta}^1_3$ everywhere. 7 is obvious. □

We are going to use the following results about models with $\mathbf{\Sigma}^1_3$-good wellorders:

Theorem 5.3 (Bagaria-Woodin)**.** *Assuming just the consistency of ZFC, there is a model, which we will denote by L^*, such that*

1. $L^* \models \mathbf{\Sigma}^1_2(\mathbb{B})$ *(and hence also* $\mathbf{\Sigma}^1_2(\mathbb{P})$ *for all* $\mathbb{P} \in \{\mathbb{C}, \mathbb{L}, \mathbb{M}, \mathbb{S}\}$*) and*
2. *there is a* $\mathbf{\Sigma}^1_3$*-good wellorder of the reals of* L^**.*

Proof. This model was first constructed in [2]. Easier constructions of models satisfying the above criteria are available using techniques developed recently by Friedman, Fischer, Zdomskyy and others (e.g. [14, 15]). □

Theorem 5.4 (David)**.** *Assuming the consistency of ZFC + inaccessible, there is a model, which we will denote by L^d, such that*

1. $L^d \models \forall r\, (\omega_1^{L[r]} < \omega_1)$*, and*
2. *there is a* $\mathbf{\Sigma}^1_3$*-good wellorder of the reals of* L^d*.*

Proof. This was proved by René David in [10]. □

We should note that the $\mathbf{\Sigma}^1_3$-good wellorder of the reals of L^* and L^d from the above results remains a $\mathbf{\Sigma}^1_3$-good wellorder (of the ground-model reals) in forcing extensions. All the models we construct will be forcing extensions of L, L^* or L^d, with the methods from the previous section guaranteeing that $\mathbf{\Delta}^1_3(\mathbb{P})$ holds for certain $\mathbb{P}$, while using Fact 5.2, together with known preservation results, to guarantee that $\mathbf{\Delta}^1_3(\mathbb{Q})$ fails for other $\mathbb{Q}$. The idea to use David's model L^d to separate regularity properties was first used by Judah and Spinas in [30].

We will use the diagrammatical notation employed by Bartoszyński and Judah in [3, Sections 7.5, 7.6], with empty circles symbolizing "false" and full circles "true". There is a total of eleven possibilities of "true"/"false"-assignments not contradicting the diagram, which we denote with the letters A–K and represent in Table 2.

In the following list we provide models for each situation. Whenever possible, the models will be constructed in ZFC alone. In three cases, namely G, H and I, we will have to make do with an inaccessible (although we conjecture that this hypothesis can be eliminated).

- **Situation A**, *determined by* $\neg\mathbf{\Delta}^1_3(\mathbb{S})$.

  This holds in L, L^* and L^d.

- **Situation B**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{S}) + \neg\mathbf{\Delta}^1_3(\mathbb{B}) + \neg\mathbf{\Delta}^1_3(\mathbb{M})$.

  The model for this is $L^{\mathbb{S}_{\omega_1}}$, i.e., the countable support iteration of Sacks forcing of length ω_1 starting from L. Since Sacks forcing is a quasi-amoeba for itself (see Example 4.4), $\mathbf{\Delta}^1_3(\mathbb{S})$ follows by Corollary 4.6. Moreover, since $\mathbb{S}_{\omega_1}$ is ω^ω-bounding and does not add random reals (by the Sacks property), it follows that, in this model, even $\mathbf{\Delta}^1_2(\mathbb{B})$ and $\mathbf{\Delta}^1_2(\mathbb{M})$ fail.

○ = FALSE ● = TRUE

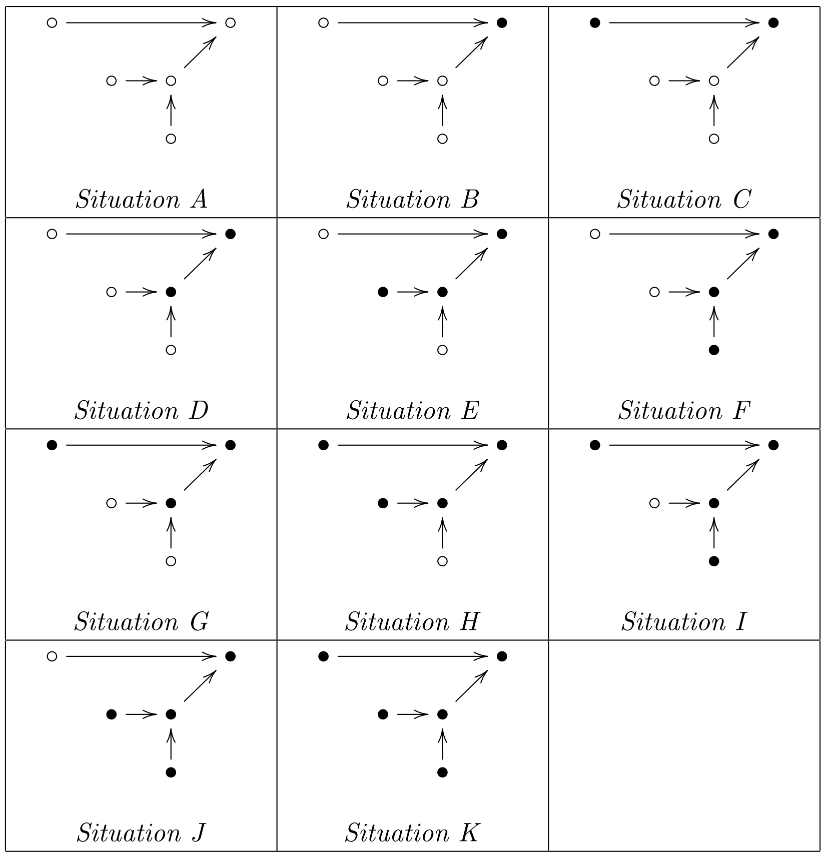

Table 2: Situations A–K in the $\mathbf{\Delta}^1_3$-diagram

- **Situation C**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{B}) + \neg\mathbf{\Delta}^1_3(\mathbb{M})$.

  Take the model $(L^*)^{\mathbb{B}_{\omega_1}}$. By Theorem 4.1 $\mathbf{\Delta}^1_3(\mathbb{B})$ holds. Because random forcing is ω^ω-bounding, and because of Fact 5.2 (6), we have $\neg\mathbf{\Delta}^1_3(\mathbb{M})$.

- **Situation D**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{M}) + \neg\mathbf{\Delta}^1_3(\mathbb{B}) + \neg\mathbf{\Delta}^1_3(\mathbb{L}) + \neg\mathbf{\Delta}^1_3(\mathbb{C})$.

  Here the model is $L^{\mathbb{M}_{\omega_1}}$. Since Miller forcing is a quasi-amoeba of itself (see Example 4.4) $\mathbf{\Delta}^1_3(\mathbb{M})$ follows by Corollary 4.6. On the other hand, Miller forcing does not add Cohen or random reals because of the Laver property ([3, Theorem 7.3.45]), so both $\mathbf{\Delta}^1_2(\mathbb{B})$ and $\mathbf{\Delta}^1_2(\mathbb{C})$ fail. Also, Miller forcing does not add dominating reals ([3, Theorem 7.3.46]), so $\mathbf{\Delta}^1_2(\mathbb{L})$ fails.

- **Situation E**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{L}) + \neg\mathbf{\Delta}^1_3(\mathbb{B}) + \neg\mathbf{\Delta}^1_3(\mathbb{C})$.

  Here, let us provide two models. Spinas, in [43], constructs a version of "amoeba for Laver" forcing, which he denotes by $\mathbb{A}(\mathbb{L})$, and proves that it is an amoeba for Laver in the sense of Definition 4.3 and, at the same time, satisfies the Laver property. It follows that the iteration $(\mathbb{L}*\mathbb{A}(\mathbb{L}))_{\omega_1}$ (i.e., the countable support iteration of length ω_1 where $\mathbb{L}$ appears at even stages and $\mathbb{A}(\mathbb{L})$ at odd stages) has the Laver property, hence $L^{(\mathbb{L}*\mathbb{A}(\mathbb{L}))_{\omega_1}} \models \neg\mathbf{\Delta}^1_2(\mathbb{B}) + \neg\mathbf{\Delta}^1_2(\mathbb{C})$. But $\mathbf{\Delta}^1_3(\mathbb{L})$ holds by Theorem 4.5.

  Another model is the one given in [29, Theorem 3.1], namely, the ω_1-iteration of Mathias forcing starting from $\mathbb{L}$. Here an even stronger assertion holds, namely "all $\mathbf{\Delta}^1_3$-sets are Ramsey" which implies $\mathbf{\Delta}^1_3(\mathbb{L})$ (see Section 6).

- **Situation F**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{C}) + \neg\mathbf{\Delta}^1_3(\mathbb{B}) + \neg\mathbf{\Delta}^1_3(\mathbb{L})$.

  Here we take $(L^*)^{\mathbb{C}_{\omega_1}}$. Then $\mathbf{\Delta}^1_3(\mathbb{C})$ holds by Theorem 4.1. On the other hand, Cohen forcing adds neither dominating nor random reals, so by Fact 5.2 (3) and (5), neither $\mathbf{\Delta}^1_3(\mathbb{B})$ nor $\mathbf{\Delta}^1_3(\mathbb{L})$ holds.

- **Situation G**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{B}) + \mathbf{\Delta}^1_3(\mathbb{M}) + \neg\mathbf{\Delta}^1_3(\mathbb{L}) + \neg\mathbf{\Delta}^1_3(\mathbb{C})$.

  Use the model $(L^d)^{(\mathbb{B}*\mathbb{M})_{\omega_1}}$, i.e., the ω_1-iteration, with countable support, of $\mathbb{B}$ and $\mathbb{M}$ interlaced starting from David's model L^d (which requires an inaccessible). By Theorem 4.10, both $\mathbf{\Delta}^1_3(\mathbb{B})$ and $\mathbf{\Delta}^1_3(\mathbb{M})$ hold. Since $\mathbb{B}$ and $\mathbb{M}$ do not add dominating reals, $\mathbf{\Delta}^1_3(\mathbb{L})$ fails by Fact 5.2 (5). To show that $\mathbf{\Delta}^1_3(\mathbb{C})$ also fails we can use a weaker version of the Laver property, namely the property of being "(F,g)-preserving" as defined in [3, Definition 7.2.23]. Both random and Miller forcing satisfy this property ([3, Lemma 7.2.25 and Theorem 7.2.26]), it is preserved in countable support iterations ([3, Theorem 7.2.29]), and it implies that no Cohen reals are added ([3, Theorem 7.2.24]). Therefore $\neg\mathbf{\Delta}^1_3(\mathbb{C})$ follows by Fact 5.2 (4).

- **Situation H**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{B}) + \mathbf{\Delta}^1_3(\mathbb{L}) + \neg\mathbf{\Delta}^1_3(\mathbb{C})$.

  Here we use $(L^d)^{(\mathbb{B}*\mathbb{L})_{\omega_1}}$, an ω_1-iteration of random and Laver forcing starting from David's model. By Theorem 4.10 $\mathbf{\Delta}^1_3(\mathbb{B})$ and $\mathbf{\Delta}^1_3(\mathbb{L})$ hold, and $\mathbf{\Delta}^1_3(\mathbb{C})$ fails for the same reason as above, namely, because both random and Laver forcing satisfy the "(F,g)-preserving" property.

- **Situation I**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{B}) + \mathbf{\Delta}^1_3(\mathbb{C}) + \neg\mathbf{\Delta}^1_3(\mathbb{L})$.

  Here we use $(L^d)^{(\mathbb{B}*\mathbb{C})_{\omega_1}}$. Again by Theorem 4.10 we have $\mathbf{\Delta}^1_3(\mathbb{B})$ and $\mathbf{\Delta}^1_3(\mathbb{C})$. But neither random nor Cohen forcing adds dominating reals, so $\neg\mathbf{\Delta}^1_3(\mathbb{L})$ fails by Fact 5.2 (5).

- **Situation J**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{L}) + \mathbf{\Delta}^1_3(\mathbb{C}) + \neg\mathbf{\Delta}^1_3(\mathbb{B})$.

  Using our methods, we can easily see that $(L^d)^{(\mathbb{C}*\mathbb{L})_{\omega_1}}$ is a model for this, where the fact that no random reals are added follows as in [3, Model 7.6.9]. However, in [26, Theorem 3.2] a model was constructed starting just from ZFC. The method there was similar to an application of our Theorem 4.5, iterating what was essentially a mixture of $\mathbb{C}$, $\mathbb{UM}$ (amoeba for category) and $\mathbb{R}_\mathcal{F}$—Mathias forcing with a Ramsey ultrafilter $\mathcal{F}$—with *finite* support, starting in L. Since the use of the Ramsey ultrafilters makes the iteration non-definable, one cannot use the arguments from Section 4 directly. Instead, the iteration was done in such a way that each segment $\mathbb{P}_{\alpha,\beta}$ of the iteration would appear again as $\dot{\mathbb{Q}}_\delta$, for cofinally many $\delta < \omega_1$ (using a bookkeeping argument like in standard MA-proofs). In [26] it was shown that such an iteration, starting from L, yields a model in which $\mathbf{\Delta}^1_3(\mathbb{R})$ (the Ramsey property; see Section 6) as well as $\mathbf{\Delta}^1_3(\mathbb{C})$ hold. The former implies $\mathbf{\Delta}^1_3(\mathbb{L})$. On the other hand, the iteration remains σ-centered implying that no random reals are added, hence $\mathbf{\Delta}^1_2(\mathbb{B})$ fails.

- **Situation K**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{B}) + \mathbf{\Delta}^1_3(\mathbb{C}) + \mathbf{\Delta}^1_3(\mathbb{L})$.

  Of course, the Solovay model satisfies this statement, so our only interest here is in constructing a model in ZFC. But this is easy: since we do not have to worry about preserving anything, we can freely apply Corollary 4.6. For example, we can use the model $L^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{UM}*\mathbb{L}*\mathbb{A}(\mathbb{L}))_{\omega_1}}$, or $L^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{UM}*\mathbb{R})_{\omega_1}}$. In fact, even $L^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{R})_{\omega_1}}$ is sufficient, because, by the Bartoszyński-Raisonnier-Stern argument, $\mathbb{A}$ already adds a comeager set of Cohen reals (cf. [3, Theorem 2.3.1]).

6 Silver and Mathias

Clearly, the techniques we developed in Section 4 are sufficiently general and can be applied to many other regularity properties related to forcing notions on the reals. In this section, we apply our techniques to two additional properties which, though not related to the cardinal numbers in Cichoń's diagram, have nevertheless received a lot of attention.

Definition 6.1.

1. A subset $A \subseteq [\omega]^\omega$ has the *Ramsey property* if $\exists a \in [\omega]^\omega$ ($[a]^\omega \subseteq A$ or $[a]^\omega \cap A = \varnothing$).
2. For $a, b \in [\omega]^\omega$ with $|b \setminus a| = \omega$, let $[a, b]^\omega := \{c \in [\omega]^\omega \mid a \subseteq c \subseteq b\}$. We call $[a, b]^\omega$ the *(a, b)-doughnut*. A subset $A \subseteq [\omega]^\omega$ has the *doughnut property* if $\exists a, b$ ($[a, b]^\omega \subseteq A$ or $[a, b]^\omega \cap A = \varnothing$).

The Ramsey property is well-known, and the doughnut property was introduced by DiPrisco and Henle in [11] as a generalization of the Ramsey property. It is not hard to see that the Ramsey and doughnut properties are equivalent to Mathias- and Silver-measurability, respectively. Therefore, we will denote them with the letters $\mathbb{R}$ and $\mathbb{V}$, which typically abbreviate the Mathias and the Silver forcing partial orders. Mathias and Silver forcing are clearly Suslin$^+$ proper.

On the $\boldsymbol{\Delta}^1_2$- and $\boldsymbol{\Sigma}^1_2$-levels, the relationship between these and other properties has been studied in [23, 20, 6]. The following are particularly interesting:

Fact 6.2 (Judah-Shelah; Halbeisen; Brendle-Halbeisen-Löwe)**.**

1. $\boldsymbol{\Sigma}^1_2(\mathbb{R}) \iff \boldsymbol{\Delta}^1_2(\mathbb{R})$.
2. $\boldsymbol{\Delta}^1_2(\mathbb{C}) \implies \boldsymbol{\Sigma}^1_2(\mathbb{V})$.
3. $\boldsymbol{\Sigma}^1_2(\mathbb{V}) \implies \boldsymbol{\Sigma}^1_2(\mathbb{M})$.

Proof. For 1 see [23, Theorem 2.7]. For 2 see [20, Lemma 2.1], and for 3 see [6, Proposition 3.5]. □

Unlike the situation with the properties we previously considered, now there are still some open questions on the second level:

Question 6.3. *Does $\boldsymbol{\Delta}^1_2(\mathbb{L}) \implies \boldsymbol{\Sigma}^1_2(\mathbb{V})$ hold? Or, at least, does $\boldsymbol{\Delta}^1_2(\mathbb{L}) \implies \boldsymbol{\Delta}^1_2(\mathbb{V})$ hold?*

As in Lemma 2.7 we have the following:

Lemma 6.4 (Folklore)**.** *Let $\boldsymbol{\Gamma}$ be closed under continuous pre-images. Then:*

1. $\boldsymbol{\Gamma}(\mathbb{R}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{V}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{S})$.
2. $\boldsymbol{\Gamma}(\mathbb{R}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{L})$.

Proof. For the first implication, note that $[a]^\omega$ is a $(\varnothing, a)$-doughnut, and the set of characteristic functions of $x \in [a,b]^\omega$ is a perfect tree in 2^ω. For the second implication, use the fact that for any $a \in [\omega]^\omega$ it is easy to find a Laver tree T such that $\forall x \in [T]\ (\mathrm{ran}(x) \subseteq a)$. □

In his PhD thesis, Laguzzi proved two additional relationships of this kind.

Lemma 6.5 (Laguzzi)**.** *Let $\boldsymbol{\Gamma}$ be closed under continuous pre-images. Then:*

1. $\boldsymbol{\Gamma}(\mathbb{C}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{V})$.
2. $\boldsymbol{\Gamma}(\mathbb{B}) \Rightarrow \boldsymbol{\Gamma}(\mathbb{V})$.

Proof. See [36, Fact 39 and Fact 55]. □

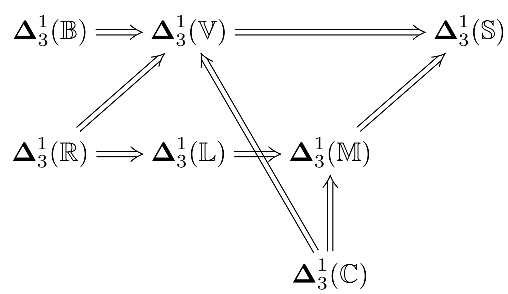

Figure 4: (Incomplete) implication diagram for $\mathbf{\Delta}^1_3$ sets of reals, including $\mathbb{R}$ and $\mathbb{V}$.

As an illustration of the application of our methods, let us repeat what we did in Section 5, i.e., look at the $\mathbf{\Delta}^1_3$-diagram with the additional properties $\mathbb{V}$ and $\mathbb{R}$ (Figure 4). There are now eighteen situations, represented in Table 3 (we have subdivided the situations from the previous section).

To find models for these situations we need the following additional facts (cf. Fact 5.2).

Definition 6.6. Let M be a model. A real $c \in [\omega]^\omega$ is

- *splitting over* M if for all $a \in M \cap [\omega]^\omega$ ($|a \cap c| = |a \setminus c| = \omega$), and
- *unsplit over* M if for all $a \in M \cap [\omega]^\omega$ ($|c \cap a| < \omega$ or $|c \setminus a| < \omega$).

Fact 6.7. *Suppose M is a model with a $\mathbf{\Sigma}^1_n$-good wellorder of the reals. Then:*

1. $\mathbf{\Delta}^1_n(\mathbb{V}) \implies \exists c\,(c$ is splitting over $M)$.
2. $\mathbf{\Delta}^1_n(\mathbb{R}) \implies \exists c\,(c$ is splitting over $M)$ *and* $\exists c\,(c$ is unsplit over $M)$.

Proof. For the first implication, use the argument in [6, Proposition 2.5], and for the second one, use [21, Theorem 2.2]. □

Splitting and unsplit reals are related to the well-known cardinal characteristics $\mathfrak{s}$ and $\mathfrak{r}$ (the *splitting* and *reaping* number, respectively), in a way similar to the relationship shown in Table 1 (although they do not *characterize* $\mathbb{R}$ and $\mathbb{V}$ in any way).

We can now find models for the following situations from Table 3 (we only list the ones that do not automatically follow from our results in Section 5).

- **Situation B1**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{S}) + \neg\mathbf{\Delta}^1_3(\mathbb{V}) + \neg\mathbf{\Delta}^1_3(\mathbb{M})$.

  Here the model is $L^{\mathbb{S}_{\omega_1}}$. Sacks forcing preserves P-points, which is an iterable property (see [17, Lemma 2.9] and [3, Theorem 6.2.6]), so in particular no splitting reals are added, hence $\mathbf{\Delta}^1_2(\mathbb{V})$ fails by Fact 6.7 (1).

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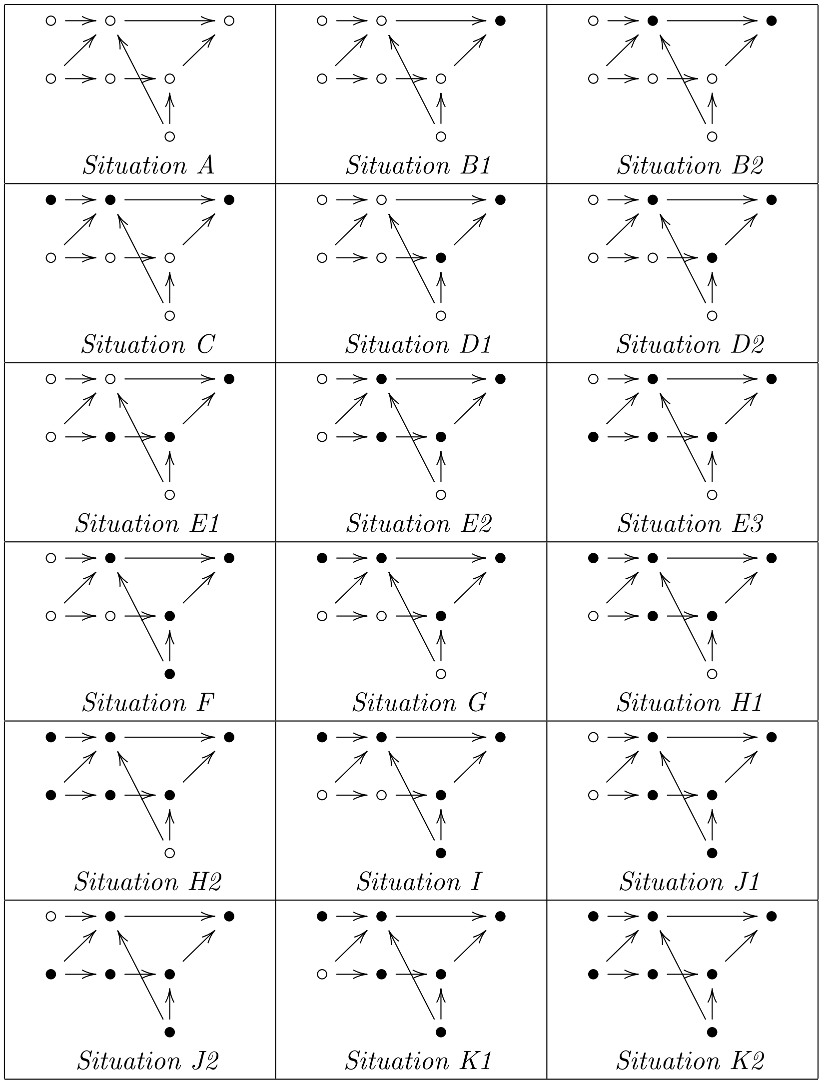

Table 3: Situations A–K in the $\mathbf{\Delta}^1_3$-diagram

- **Situation B2**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{V}) + \neg\mathbf{\Delta}^1_3(\mathbb{B}) + \neg\mathbf{\Delta}^1_3(\mathbb{M})$.

  Use $(L^d)^{\mathbb{V}_{\omega_1}}$. By the Sacks property of Silver forcing neither random nor unbounded reals are added.

- **Situation D1**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{M}) + \neg\mathbf{\Delta}^1_3(\mathbb{V}) + \neg\mathbf{\Delta}^1_3(\mathbb{L}) + \neg\mathbf{\Delta}^1_3(\mathbb{C})$.

Here we use $L^{\mathbb{M}_{\omega_1}}$. Again $\mathbf{\Delta}^1_2(\mathbb{V})$ fails because Miller forcing preserves P-points [3, Lemma 7.3.48].

- **Situation D2**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{V}) + \mathbf{\Delta}^1_3(\mathbb{M}) + \neg\mathbf{\Delta}^1_3(\mathbb{B}) + \neg\mathbf{\Delta}^1_3(\mathbb{L}) + \neg\mathbf{\Delta}^1_3(\mathbb{C})$.

  Here we can use $(L^d)^{(\mathbb{V}*\mathbb{M})_{\omega_1}}$. Both $\mathbb{V}$ and $\mathbb{M}$ have the Laver property, and both do not add dominating reals.

- **Situation E1**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{L}) + \neg\mathbf{\Delta}^1_3(\mathbb{V})$.

  We don't know if this situation is consistent!

- **Situation E2**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{L}) + \mathbf{\Delta}^1_3(\mathbb{V}) + \neg\mathbf{\Delta}^1_3(\mathbb{B}) + \neg\mathbf{\Delta}^1_3(\mathbb{R}) + \neg\mathbf{\Delta}^1_3(\mathbb{C})$.

  Use $(L^d)^{(\mathbb{L}*\mathbb{V})_{\omega_1}}$. Both $\mathbb{L}$ and $\mathbb{V}$ have the Laver property, implying that neither random nor Cohen reals are added. To show that $\mathbf{\Delta}^1_3(\mathbb{R})$ fails, recall the preservation property called "preserving $\sqsubseteq^{\text{random}}$" in [3, Definition 6.3.7]. Both $\mathbb{L}$ and $\mathbb{V}$ satisfy this property (for $\mathbb{L}$ see [3, Theorem 7.3.39] and for $\mathbb{V}$ it follows from an even stronger result, namely [3, Lemma 6.3.39]), it is preserved by countable support iteration [3, Theorem 6.1.13] and implies that the ground model reals have positive measure [3, Thorem 6.3.13]. From this, one can infer that there are no unsplit reals over the ground model, in a way analogous to the well-known proof of the cardinal inequality $\mathfrak{s} \leq \text{non}(\mathcal{N})$ (i.e., for every $a \in [\omega]^\omega$, the set $X_a := \{b \mid b$ does not split $a\}$ has measure zero). Hence, the result follows from Fact 6.7 (2).

- **Situation E3**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{R}) + \neg\mathbf{\Delta}^1_3(\mathbb{B}) + \neg\mathbf{\Delta}^1_3(\mathbb{C})$.

  Clearly $L^{\mathbb{R}_{\omega_1}}$ works here.

- **Situation H1**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{B}) + \mathbf{\Delta}^1_3(\mathbb{L}) + \neg\mathbf{\Delta}^1_3(\mathbb{R}) + \neg\mathbf{\Delta}^1_3(\mathbb{C})$.

  Use $(L^d)^{(\mathbb{B}*\mathbb{L})_{\omega_1}}$. Both $\mathbb{B}$ and $\mathbb{L}$ have the "(F, g)-preserving" property, implying that no Cohen reals are added. To show that $\mathbf{\Delta}^1_3(\mathbb{R})$ fails use again the "preserving $\sqsubseteq^{\text{random}}$"-property. Random forcing satisfies this by [3, Lemma 6.3.12], so, as before, we are done by Fact 6.7 (2).

- **Situation H2**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{B}) + \mathbf{\Delta}^1_3(\mathbb{R}) + \neg\mathbf{\Delta}^1_3(\mathbb{C})$.

  Use $(L^d)^{(\mathbb{B}*\mathbb{R})_{\omega_1}}$. Both $\mathbb{B}$ and $\mathbb{R}$ have the "(F, g)-preserving" property, implying that no Cohen reals are added.

- **Situation J1**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{C}) + \mathbf{\Delta}^1_3(\mathbb{L}) + \neg\mathbf{\Delta}^1_3(\mathbb{B}) + \neg\mathbf{\Delta}^1_3(\mathbb{R})$.

  Use $(L^d)^{(\mathbb{C}*\mathbb{L})_{\omega_1}}$. As in [3, Model 7.6.9] we can show that no random reals are added by the iteration. To show that $\mathbf{\Delta}^1_3(\mathbb{R})$ fails, we note that both $\mathbb{C}$ and $\mathbb{L}$ satisfy a strong iterable property implying that no unsplit reals are added: see e.g. [12, Lemma 8, 9] and [5, Main Lemma 1.11], and apply Fact 6.7 (2).

- **Situation J2**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{C}) + \mathbf{\Delta}^1_3(\mathbb{R}) + \neg\mathbf{\Delta}^1_3(\mathbb{B})$.

  Here we can either use $(L^d)^{(\mathbb{C}*\mathbb{R})}$ or the ZFC-model from [26, Theorem 3.2] which we also used in Situation J in Section 5.

- **Situation K1**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{B}) + \mathbf{\Delta}^1_3(\mathbb{L}) + \mathbf{\Delta}^1_3(\mathbb{C}) + \neg\mathbf{\Delta}^1_3(\mathbb{R})$.

  Use $(L^d)^{(\mathbb{B}*\mathbb{L}*\mathbb{C})_{\omega_1}}$. To show that no unsplit reals are added, use the iterable version for $\mathbb{C}$ and $\mathbb{L}$ (as in Situation J1) and preservation of $\sqsubseteq^{\text{random}}$ for $\mathbb{B}$. Again, $\mathbf{\Delta}^1_3(\mathbb{R})$ fails by Fact 6.7 (2).

- **Situation K2**, *determined by* $\mathbf{\Delta}^1_3(\mathbb{B}) + \mathbf{\Delta}^1_3(\mathbb{R}) + \mathbf{\Delta}^1_3(\mathbb{C})$.

  Here $L^{(\mathbb{B}*\mathbb{A}*\mathbb{R}*\mathbb{C})_{\omega_1}}$ clearly suffices.

To conclude: all situations except E1 are consistent. Moreover, we have ZFC-models for B1, D1, E3, J2 and K2, whereas for the other cases we need an inaccessible. The difficulty concerning Situation E1 lies in the fact that we do not know whether $\mathbf{\Delta}^1_3(\mathbb{L}) \Rightarrow \mathbf{\Delta}^1_3(\mathbb{V})$ holds (cf. Question 6.3).

7 Beyond $\mathbf{\Delta}^1_3$

Although our techniques were primarily developed to deal with the $\mathbf{\Delta}^1_3$-level of the projective hieararchy, there are some applications to higher levels as well. In this section we summarize what can be said about higher levels using our techniques. We have two applications: concerning the diagram on the $\mathbf{\Delta}^1_4$-level, and concerning the separation of $\mathbf{\Delta}^1_3$-regularity from $\mathbf{\Sigma}^1_3$-regularity, as well as $\mathbf{\Delta}^1_4$-regularity from $\mathbf{\Sigma}^1_4$-regularity.

7.1 The $\mathbf{\Delta}^1_4$-diagram

Consider the analogue of Figure 3 but with $\mathbf{\Delta}^1_3$ replaced by $\mathbf{\Delta}^1_4$. It turns out that, under the assumption $\forall r\ (\omega_1^{L[r]} < \omega_1)$, both Theorem 4.1 and Theorem 4.8 have suitable generalizations. The first generalization is due to Judah and Spinas:

Theorem 7.1 (Judah-Spinas)**.** *Assuming* $\forall r\,(\omega_1^{L[r]} < \omega_1)$*, there exists a model* N_0*, which has a* $\mathbf{\Sigma}^1_4$*-good wellorder of the reals, and, moreover, such that* $N_0^{\mathbb{B}_{\omega_1}} \models \mathbf{\Delta}^1_4(\mathbb{B})$ *and* $N_0^{\mathbb{C}_{\omega_1}} \models \mathbf{\Delta}^1_4(\mathbb{C})$*.*

For the proof, see [30]. The method is, in essence, an analogue of Theorem 4.1, but starting from David's model L^d instead of L, and using some additional tricks.

The following is a direct generalization of our own Theorem 4.5.

Theorem 7.2. *Suppose* $V \models \forall r\,(\omega_1^{L[r]} < \omega_1)$*,* $\mathbb{P}$ *is a real forcing notion and* $\mathbb{AP}$ *a quasi-amoeba for* $\mathbb{P}$*, and assume that both* $\mathbb{P}$ *and* $\mathbb{AP}$ *are Suslin*$^+$ *proper. Let* $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ *be a Suslin*$^+$ *proper iteration whose iterands are* $\mathbb{P}$ *and* $\mathbb{AP}$ *interlaced* (*i.e., the same conditions hold as in Theorem 4.5*)*. Then* $V^{\mathbb{P}_{\omega_1}} \models \mathbf{\Delta}^1_4(\mathbb{P})$*.*

Proof. The proof is exactly the same as that of Theorem 4.5. The reader can verify that every step in that proof is valid if we:

1. Replace $\mathbf{\Delta}^1_3$ by $\mathbf{\Delta}^1_4$, $\mathbf{\Sigma}^1_3$ by $\mathbf{\Sigma}^1_4$ and $\mathbf{\Pi}^1_2$ by $\mathbf{\Pi}^1_3$ everywhere.
2. Use $\mathbf{\Sigma}^1_3$-$\mathbb{P}_{\omega_1}$-correctness instead of Shoenfield absoluteness everywhere, which is valid by Theorem 3.20. Notice that in the proof we only used Shoenfield absoluteness between models that lay between V and $V[G_{\omega_1}]$, so by Fact 3.19 we are safe.
3. Use Lemma 3.14 (4) to conclude that $\tilde{\theta}$ is $\mathbf{\Pi}^1_3$.

□

Just as before, we actually have a stronger version which allows us to mix different partial orders $\mathbb{P}$.

Theorem 7.3. *Suppose* $V \models \forall r\,(\omega_1^{L[r]} < \omega_1)$*,* $\mathbb{P}$ *and* $\mathbb{AP}$ *are as before,* $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ *is Suslin*$^+$ *proper with parameters in the ground model and repetitive, and both* $\mathbb{P}$ *and* $\mathbb{AP}$ *appear cofinally often in the iteration. Then* $V^{\mathbb{P}_{\omega_1}} \models \mathbf{\Delta}^1_4(\mathbb{P})$*.*

Using Theorem 7.1 and Theorem 7.3, it follows that we can construct a model for any of the situations from Section 5 which was obtained by an application of the first two methods (Theorem 4.1 and Theorem 4.8) and *not* by tdhe third method (Theorem 4.10).

Corollary 7.4. *Situations A, B, C, D, E, F and K in the* $\mathbf{\Delta}^1_4$*-diagram are consistent relative to ZFC + inaccessible.*

7.2 Separating Δ from Σ

Recall that, in the long-run, we would like to find "complete" diagrams on the combined $\mathbf{\Delta}^1_n$- and $\mathbf{\Sigma}^1_n$-levels, for $n \geq 3$ (cf. Figure 2). But there are many obstacles, and the most urgent one seems to be the following:

Question 7.5. *Does $\mathbf{\Sigma}^1_n(\mathbb{P}) \iff \mathbf{\Delta}^1_n(\mathbb{P})$ hold for $\mathbb{P} \in \{\mathbb{L}, \mathbb{M}, \mathbb{S}\}$ and $n \geq 3$?*

Further progress in the study of the joint $\mathbf{\Sigma}^1_n/\mathbf{\Delta}^1_n$-diagram seems to depend largely on the solution to the above question.

Nevertheless, there are a few interesting things we can prove. Recall that, after Shelah proved that $\mathbf{\Sigma}^1_3(\mathbb{B})$ implies an inaccessible in [40], Raisonnier [39] provided an alternative and simpler proof, based on the following:

Definition 7.6. Let $\mathcal{F}$ be a non-principal filter on ω. $\mathcal{F}$ is called a *rapid filter* if

$$\forall c \in [\omega]^\omega \, \exists a \in \mathcal{F} \, \forall n \, (|c(n) \cap a| \leq n),$$

or, equivalently, if $\mathcal{F}$ considered as a subset of $\omega^{\uparrow\omega}$ (the space of strictly increasing functions from ω to ω) is a dominating family in $\omega^{\uparrow\omega}$.

The point is that rapid filters provide natural counterexamples to several regularity properties. The following is a folklore result:

Fact 7.7. *If $\mathcal{F}$ is a rapid filter, then $\mathcal{F}$ (considered as a subset of $\omega^{\uparrow\omega}$) is not measurable and does not have the Baire property.*

Rapid filters also provide counterexamples to the Ramsey and doughnut properties from Section 6, albeit using a derived construction:

Definition 7.8. For $a \in [\omega]^\omega$, let

$$\ddot{a} := [0, a(0)) \,\cup\, [a(1), a(2)) \,\cup\, [a(3), a(4)) \,\cup\, \dots$$

where $\{a(n) \mid n < \omega\}$ is the increasing enumeration of a. If $\mathcal{F}$ is a filter on ω, let $\ddot{\mathcal{F}} := \{a \in [\omega]^\omega \mid \ddot{a} \in \mathcal{F}\}$.

It is clear that the operation $\mathcal{F} \mapsto \ddot{\mathcal{F}}$ does not increase the complexity. In [37] Mathias proved that if $\mathcal{F}$ is a rapid filter then $\ddot{\mathcal{F}}$ does not have the Ramsey property. We improve this result as follows:

Lemma 7.9. *If $\mathcal{F}$ is a rapid filter then $\ddot{\mathcal{F}}$ does not have the doughnut property.*

Proof. Let $a, b \in [\omega]^\omega$ be arbitrary and assume $|b \setminus a| = \omega$. It is easy to see that $[a, b]^\omega$ cannot be a subset of $\ddot{\mathcal{F}}$: pick any $x, y \in [a, b]^\omega$ such that $x = y \setminus \{n\}$ for some n. Then, clearly, $\ddot{x}$ and $\ddot{y}$ have finite intersection, so x and y cannot both be in $\ddot{\mathcal{F}}$ (this argument works for any non-principal filter $\mathcal{F}$).

So it remains to show that $[a, b]^\omega$ cannot be completely disjoint from $\ddot{\mathcal{F}}$. Let f be an enumeration of $b \setminus a$. As $\mathcal{F}$ is rapid, there is a $y \in \mathcal{F}$ be such that for all n, $|f(n) \cap y| \leq n$. We will find an $x \in [a, b]^\omega \,\cap\, \ddot{\mathcal{F}}$. The real x is constructed as follow: if $i \in a$ then $i \in x$; if $i \notin b$ then $i \notin x$; and if $i \in b \setminus a$, then, whether i is in x or not will depends on the consideration described below (notice that, in any case, x will be a member of $[a, b]^\omega$). For every $n \geq 1$ and every element $y(n)$, there is always at least one member of $b \setminus a$ which lies strictly between $y(n-1)$ and $y(n)$. Let m_n be the largest of them. Now it is easy to see that by making the right choice of either "$m_n \in x$" or "$m_n \notin x$" we can always make sure that $y(n)$ is in $\ddot{x} = [0, x(0)) \cup [x(1), x(2)) \cup \ldots$ (it does not matter what we do with the other $i \in b \setminus a$ which lie between $y(n-1)$ and $y(n)$). If we do this for every n, we obtain a set x which is in $[a, b]^\omega$, and moreover, $y \setminus \{y(0)\} \subseteq \ddot{x}$. Since $y \in \mathcal{F}$ holds by assumption, $x \in \ddot{\mathcal{F}}$ follows. □

Raisonnier's proof of Shelah's theorem is based on the following crucial lemma:

Lemma 7.10 (Raisonnier)**.** *Suppose $\omega_1^L = \omega_1$ and $\mathbf{\Sigma}^1_2(\mathbb{B})$ holds. Then there exists a $\mathbf{\Sigma}^1_3$ rapid filter (the* Raisonnier filter*).*

Looking at Raisonnier's argument, it is straightforward to obtain the following generalization to higher projective levels:

Lemma 7.11. *Suppose M is a model with a $\mathbf{\Sigma}^1_n$-good wellorder of the reals. If $\omega_1^M = \omega_1$ and for every r there is a measure-one set of random reals over $M[r]$, then there exists a $\mathbf{\Sigma}^1_{n+1}$ rapid filter.*

We can use Raisonnier's argument to prove the following separation results:

Theorem 7.12.

1. *It is consistent relative to ZFC that $\mathbf{\Delta}^1_3(\mathbb{P})$ holds for all $\mathbb{P}$ considered in Sections 5 and 6, but $\mathbf{\Sigma}^1_3(\mathbb{B}), \mathbf{\Sigma}^1_3(\mathbb{C}), \mathbf{\Sigma}^1_3(\mathbb{R})$ and $\mathbf{\Sigma}^1_3(\mathbb{V})$ fail.*

2. *It is consistent relative to ZFC + inaccessible that $\mathbf{\Delta}^1_4(\mathbb{P})$ holds for all $\mathbb{P}$ considered in Sections 5 and 6, but $\mathbf{\Sigma}^1_4(\mathbb{B}), \mathbf{\Sigma}^1_4(\mathbb{C}), \mathbf{\Sigma}^1_4(\mathbb{R})$ and $\mathbf{\Sigma}^1_4(\mathbb{V})$ fail.*

Proof. For 1, take the model for Situation K2 in Section 6, i.e., $L^{(\mathbb{B}*\mathbb{A}*\mathbb{R}*\mathbb{C})_{\omega_1}}$. Since both antecedents of Lemma 7.10 are satisfied we are done by Fact 7.7 and Lemma 7.9. For 2, take the model $(L^d)^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{R})_{\omega_1}}$, use Theorem 7.3 to obtain $\mathbf{\Delta}^1_4(\mathbb{P})$ for all $\mathbb{P}$, and again note that both antecedents of Lemma 7.11 are satisfied for $M = L^d$ and $n = 3$, so again we are done by Fact 7.7 and Lemma 7.9. □

The above result has a number of interesting consequences regarding "non-lifting" of implications that were true on the second level. We had the following non-trivial implications:

1. $\mathbf{\Delta}^1_2(\mathbb{C}) + \mathbf{\Delta}^1_2(\mathbb{L}) \Rightarrow \mathbf{\Sigma}^1_2(\mathbb{C})$ (analogue of Truss, cf. [45]).
2. $\mathbf{\Delta}^1_2(\mathbb{R}) \implies \mathbf{\Sigma}^1_2(\mathbb{R})$ (Judah-Shelah, cf. [23]).
3. $\mathbf{\Delta}^1_2(\mathbb{C}) \implies \mathbf{\Sigma}^1_2(\mathbb{V})$ (Halbeisen, cf. [20, Lemma 2.1]).

By Theorem 7.12, all of the above fail to lift to the third and fourth levels of the projective hierarchy.

8 Open questions

Although we have made significant progress in this area of research, many questions are still open. The most urgent question seems to be:

Question 8.1. *Is* $\mathbf{\Delta}^1_3(\mathbb{P}) + \neg\mathbf{\Sigma}^1_3(\mathbb{P})$ *consistent for* $\mathbb{P} \in \{\mathbb{S}, \mathbb{M}, \mathbb{L}\}$*? More generally, is* $\mathbf{\Delta}^1_n(\mathbb{P}) + \neg\mathbf{\Sigma}^1_n(\mathbb{P})$ *consistent for these* $\mathbb{P}$*?*

We conjecture that the answer is positive. Recall that in Theorem 7.12 we proved the consistency of $\mathbf{\Delta}^1_3(\mathbb{R}) + \neg\mathbf{\Sigma}^1_3(\mathbb{R})$ using the Raisonnier filter. It would seem plausible that a similar method will work to settle Question 8.1 as well. In fact, we conjecture the following:

Conjecture 8.2. *It is consistent, relative to ZFC, that* $\mathbf{\Delta}^1_3(\mathbb{P})$ *holds for all* $\mathbb{P}$ *but* $\mathbf{\Sigma}^1_3(\mathbb{S})$ *fails (and therefore,* $\mathbf{\Sigma}^1_3(\mathbb{P})$ *fails for all* $\mathbb{P}$*).*

The next question concerns the use of inaccessibles in our proofs. Of course, when proving results about $\mathbf{\Sigma}^1_3(\mathbb{P})$ or higher projective sets, inaccessibles cannot be avoided (at least if our proofs are to work uniformly for all $\mathbb{P}$). However, they are not necessary for $\mathbf{\Delta}^1_3$-results, and their use in our proofs seems to arise mostly from a lack of finer methods. Therefore we conjecture the following:

Conjecture 8.3. *All the situations on the* $\mathbf{\Delta}^1_3$*-level (specifically Situations* G, H, *and* I *from Section 5 and* B2, D2, E2, H1, H2, J1 *and* K1 *from Section 6) have models based just in ZFC.*

The plan would be to improve Theorem 4.10 by replacing the assumption $\forall r\, (\omega_1^{L[r]} < \omega_1)$ by a weaker assumption (for example, about the existence of many generics over $L[r]$) that can be obtained without inaccessibles but is still sufficiently strong to guarantee similar results. Then we can obtain models using this method, starting with some other ZFC-model instead of L^d.

Other questions involve finding complete diagrams for levels beyond $\mathbf{\Delta}^1_3$.

Question 8.4.

1. *Find a complete diagram for* $\mathbf{\Delta}^1_3$*- and* $\mathbf{\Sigma}^1_3$*-regularity (cf. Figure 2).*
2. *Find models for Situations G, H, I and J in the* $\mathbf{\Delta}^1_4$*-diagram.*
3. *Find a complete diagram for regularity properties of all projective sets.*

Finally, the following simple questions are well-known, but have, so far, remained unresolved:

Question 8.5. *What is the consistency strength of* $\mathbf{\Sigma}^1_3(\mathbb{R})$ *and* $\mathbf{\Sigma}^1_3(\mathbb{L})$*?*

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References

[1] Joan Bagaria and Sy D. Friedman. Generic absoluteness. In *Proceedings of the XIth Latin American Symposium on Mathematical Logic (Mérida, 1998)*, volume 108, pages 3–13, 2001.

[2] Joan Bagaria and W. Hugh Woodin. $\boldsymbol{\Delta}^1_n$ sets of reals. *J. Symbolic Logic*, 62(4):1379–1428, 1997.

[3] Tomek Bartoszyński and Haim Judah. *Set Theory, On the Structure of the Real Line*. A K Peters, 1995.

[4] Jörg Brendle. How small can the set of generics be? In *Logic Colloquium '98 (Prague)*, volume 13 of *Lect. Notes Log.*, pages 109–126. Assoc. Symbol. Logic, Urbana, IL, 2000.

[5] Jörg Brendle. Forcing and the structure of the real line: the Bogotá lectures. Lecture notes, 2009.

[6] Jörg Brendle, Lorenz Halbeisen, and Benedikt Löwe. Silver measurability and its relation to other regularity properties. *Math. Proc. Cambridge Philos. Soc.*, 138(1):135–149, 2005.

[7] Jörg Brendle and Yurii Khomskii. Polarized partitions on the second level of the projective hierarchy. *Ann. Pure Appl. Logic*, 163(9):1345–1357, 2012.

[8] Jörg Brendle and Benedikt Löwe. Solovay-type characterizations for forcing-algebras. *J. Symbolic Logic*, 64(3):1307–1323, 1999.

[9] Jörg Brendle and Benedikt Löwe. Eventually different functions and inaccessible cardinals. *J. Math. Soc. Japan*, 63(1):137–151, 2011.

[10] René David. Δ^1_3 reals. *Ann. Math. Logic*, 23(2-3):121–125 (1983), 1982.

[11] Carlos A. Di Prisco and James M. Henle. Doughnuts, floating ordinals, square brackets, and ultraflitters. *J. Symbolic Logic*, 65(1):461–473, 2000.

[12] Alan Dow. Two classes of Fréchet-Urysohn spaces. *Proc. Amer. Math. Soc.*, 108(1):241–247, 1990.

[13] Qi Feng, Menachem Magidor, and Hugh Woodin. Universally Baire sets of reals. In *Set theory of the continuum (Berkeley, CA, 1989)*, volume 26 of *Math. Sci. Res. Inst. Publ.*, pages 203–242. Springer, New York, 1992.

[14] Vera Fischer and Sy David Friedman. Cardinal characteristics and projective wellorders. *Ann. Pure Appl. Logic*, 161(7):916–922, 2010.

[15] Vera Fischer, Sy David Friedman, and Lyubomyr Zdomskyy. Projective wellorders and mad families with large continuum. *Annals of Pure and Applied Logic*, 162(11):853 – 862, 2011.

[16] Sy-David Friedman and David Schrittesser. Projective measure without projective Baire. Preprint, March 2013.

[17] Stefan Geschke and Sandra Quickert. On Sacks forcing and the Sacks property. In *Classical and new paradigms of computation and their complexity hierarchies*, volume 23 of *Trends Log. Stud. Log. Libr.*, pages 95–139. Kluwer Acad. Publ., Dordrecht, 2004.

[18] M. Goldstern and H. Judah. Iteration of Souslin forcing, projective measurability and the Borel conjecture. *Israel J. Math.*, 78(2-3):335–362, 1992.

[19] Martin Goldstern. Tools for your forcing construction. In *Set theory of the reals (Ramat Gan, 1991)*, volume 6 of *Israel Math. Conf. Proc.*, pages 305–360. Bar-Ilan Univ., Ramat Gan, 1993.

[20] Lorenz Halbeisen. Making doughnuts of Cohen reals. *Math. Logic Q.*, 49(2):173–178, 2003.

[21] Jaime I. Ihoda. Σ^1_2-sets of reals. *J. Symbolic Logic*, 53(2):636–642, 1988.

[22] Jaime I. Ihoda and Saharon Shelah. Souslin forcing. *J. Symbolic Logic*, 53(4):1188–1207, 1988.

[23] Jaime I. Ihoda and Saharon Shelah. Δ^1_2-sets of reals. *Ann. Pure Appl. Logic*, 42(3):207–223, 1989.

[24] Daisuke Ikegami. Forcing absoluteness and regularity properties. *Ann. Pure Appl. Logic*, 161(7):879–894, 2010.

[25] R. B. Jensen and R. M. Solovay. Some applications of almost disjoint sets. In *Mathematical Logic and Foundations of Set Theory (Proc. Internat. Colloq., Jerusalem, 1968)*, pages 84–104. North-Holland, Amsterdam, 1970.

[26] Haim Judah. Exact equiconsistency results for $\boldsymbol{\Delta^1_3}$-sets of reals. *Arch. Math. Logic*, 32(2):101–112, 1992.

[27] Haim Judah. Absoluteness for projective sets. In *Logic Colloquium '90 (Helsinki, 1990)*, volume 2 of *Lecture Notes Logic*, pages 145–154. Springer, Berlin, 1993.

[28] Haim Judah. Δ^1_3-sets of reals. In *Set theory of the reals (Ramat Gan, 1991)*, volume 6 of *Israel Math. Conf. Proc.*, pages 361–384. Bar-Ilan Univ., Ramat Gan, 1993.

[29] Haim Judah and Saharon Shelah. $\boldsymbol{\Delta}^1_3$-sets of reals. *J. Symbolic Logic*, 58(1):72–80, 1993.

[30] Haim Judah and Otmar Spinas. On the structure of $\boldsymbol{\Delta}^1_4$-sets of reals. *Arch. Math. Logic*, 34(5):301–312, 1995.

[31] Haim Judah and Otmar Spinas. Large cardinals and projective sets. *Arch. Math. Logic*, 36(2):137–155, 1997.

[32] V. G. Kanoveĭ and V. A. Lyubetskiĭ. On some classical problems in descriptive set theory. *Uspekhi Mat. Nauk*, 58(5(353)):3–88, 2003.

[33] Jakob Kellner. Preserving non-null with Suslin$^+$ forcings. *Arch. Math. Logic*, 45(6):649–664, 2006.

[34] Jakob Kellner. Non elementary proper forcing. Preprint, 2013.

[35] Yurii Khomskii. *Regularity Properties and Definability in the Real Number Continuum.* PhD thesis, University of Amsterdam, 2012. ILLC Dissertations DS-2012-04.

[36] Giorgio Laguzzi. *Arboreal Forcing Notions and Regularity Properties of the Real Line.* PhD thesis, University of Vienna, 2012.

[37] A. R. D. Mathias. A remark on rare filters. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdös on his 60th birthday), Vol. III*, pages 1095–1097. Colloq. Math. Soc. János Bolyai, Vol. 10. North-Holland, Amsterdam, 1975.

[38] Adrian R. D. Mathias. Happy families. *Ann. Math. Logic*, 12(1):59–111, 1977.

[39] Jean Raisonnier. A mathematical proof of S. Shelah's theorem on the measure problem and related results. *Israel J. Math.*, 48(1):48–56, 1984.

[40] Saharon Shelah. Can you take Solovay's inaccessible away? *Israel J. Math.*, 48(1):1–47, 1984.

[41] Saharon Shelah. Properness without elementaricity. *J. Appl. Anal.*, 10(2):169–289, 2004.

[42] Robert M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. of Math. (2)*, 92:1–56, 1970.

[43] Otmar Spinas. Generic trees. *J. Symbolic Logic*, 60(3):705–726, 1995.

[44] Edward Szpilrajn. Sur une classe de fonctions de M. Sierpiski et la classe correspondante d'ensembles. *Fund. Math.*, 24:17–14, 1935.

[45] John Truss. Sets having calibre $\aleph_1$. In *Logic Colloquium 76 (Oxford, 1976)*, pages 595–612. Studies in Logic and Found. Math., Vol. 87. North-Holland, Amsterdam, 1977.

[46] W. Hugh Woodin. On the consistency strength of projective uniformization. In *Proceedings of the Herbrand symposium (Marseilles, 1981)*, volume 107 of *Stud. Logic Found. Math.*, pages 365–384, Amsterdam, 1982. North-Holland.

[47] Jindřich Zapletal. *Forcing idealized*, volume 174 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2008.