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MAD FAMILIES CONSTRUCTED FROM PERFECT ALMOST DISJOINT FAMILIES

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Abstract. We prove the consistency of $\mathfrak{b} > \aleph_1$ together with the existence of a $\mathbf{\Pi}^1_1$-definable mad family, answering a question posed by Friedman and Zdomskyy in [7, Question 16]. For the proof we construct a mad family in L which is an $\aleph_1$-union of perfect a.d. sets, such that this union remains mad in the iterated Hechler extension. The construction also leads us to isolate a new cardinal invariant, the *Borel almost-disjointness number* $\mathfrak{a}_B$, defined as the least number of Borel a.d. sets whose union is a mad family. Our proof yields the consistency of $\mathfrak{a}_B < \mathfrak{b}$ (and hence, $\mathfrak{a}_B < \mathfrak{a}$).

§1. Introduction.

A family A of infinite subsets of ω is called *almost disjoint* (*a.d.*) if any two elements a, b of A have finite intersection. A family A is called *maximal almost disjoint*, or *mad*, if it is an infinite a.d. family which is maximal with respect to that property—in other words, $\forall a\, \exists b \in A\, (|a \cap b| = \omega)$. The starting point of this paper is the following theorem of Adrian Mathias [11, Corollary 4.7]:

THEOREM 1.1 (Mathias). *There are no analytic mad families.*

On the other hand, it is easy to see that in L there is a $\mathbf{\Sigma}^1_2$ definable mad family. In [12, Theorem 8.23], Arnold Miller used a sophisticated method to prove the seemingly stronger result that in L there is a $\mathbf{\Pi}^1_1$ definable mad family. However, by a recent result of Asger Törnquist [15] it turns out that the existence of a $\mathbf{\Pi}^1_1$ mad family is equivalent to the existence of a $\mathbf{\Sigma}^1_2$ mad family.

By well-known results (see e.g. [10, 8, 4]), one can construct mad families which remain mad in iterations of some standard forcing notions, among which Cohen, random, Sacks and Miller forcing; such families are called $\mathbb{P}$-*indestructible*, where $\mathbb{P}$ is the forcing notion in question. Since these constructions proceed via recursion on a well-ordering of the reals, it is clear that in L one

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can construct $\mathbb{P}$-indestructible mad families that have $\mathbf{\Sigma}^1_2$ definitions. If $A \in L$ is such a $\mathbb{P}$-indestuctible mad family, with defining $\mathbf{\Sigma}^1_2$ formula $\phi(x)$, then in the iterated extension of L via $\mathbb{P}$ the family A is still mad, and moreover still has a $\mathbf{\Sigma}^1_2$ definition, namely with defining formula $\psi(x) \equiv \phi(x) \wedge x \in L$. It follows that the existence of a $\mathbf{\Sigma}^1_2$ mad family is consistent with $\neg$CH, and by Törnquist's result, so is the existence of a $\mathbf{\Pi}^1_1$ mad family. If we use the Cohen or random iterations in the preceding argument, we can even improve $\neg$CH to the stronger assertions that $\mathrm{cov}(\mathcal{M}) > \aleph_1$ or $\mathrm{cov}(\mathcal{N}) > \aleph_1$, respectively.

What can be said of forcing extensions that have stronger transcendence properties? In particular, what if they add dominating reals? Recall the cardinal invariant $\mathfrak{a}$, defined as the smallest size of an (infinite) mad family, and $\mathfrak{b}$, the smallest size of an unbounded family. It is known that $\mathfrak{b} \leq \mathfrak{a}$ and, in fact, the proof of this inequality tells us that if A is a mad family and $\mathbb{P}$ adds a dominating real, then in the generic extension $V[G]$ by $\mathbb{P}$, A is no longer a mad family—in other words, there are no $\mathbb{P}$-indestructible mad families for forcings $\mathbb{P}$ which add a dominating real.

This raises the following question: can we iterate a forcing that adds dominating reals, and still have a $\mathbf{\Pi}^1_1$ mad family in the extension? So, is the existence of a $\mathbf{\Pi}^1_1$ mad family consistent with $\mathfrak{b} > \aleph_1$? Note that the method used so far (i.e., constructing a definable mad family in L and preserving it) cannot work here.

In a recent result, Friedman and Zdomskyy proved the following result [7, Theorem 1]:

Theorem 1.2 (Friedman & Zdomskyy). *It is consistent that* $\mathfrak{b} > \aleph_1$ *and there exists a* $\mathbf{\Pi}^1_2$ *ω-mad family.*

This was further extended in [6], where the existence of a $\mathbf{\Pi}^1_2$ ω-mad family was shown to be consistent with $\mathfrak{b} = \aleph_3$ and the existence of a Δ^1_3-definable well-order of the reals. Here, an *ω-mad* family is a mad family satisfying a stronger maximality requirement (see e.g. [9] for a definition). Theorem 1.2 is optimal for ω-mad families: if A were a $\mathbf{\Sigma}^1_2$ ω-mad family, then it would either be a subset of L or would contain a perfect set. The former is false because $\aleph_1 < \mathfrak{b} \leq \mathfrak{a}$ and the latter is impossible because an ω-mad family cannot contain a perfect set by [13, Corollary 38]. Friedman and Zdomskyy [7, Question 16] asked whether a better result is possible for the more general case of a mad family. We answer this question positively.

Main Theorem 1. *It is consistent that* $\mathfrak{b} > \aleph_1$ *and there exists a* $\mathbf{\Pi}^1_1$ *mad family.*

To avoid the problem with dominating reals we need a somewhat new approach to *preservation*: rather than constructing a mad family A whose maximality is preserved directly, we construct A as a union of $\aleph_1$-many Borel sets in such a way that the union of the same sets *re-interpreted in the larger model* remains a mad family.

Definition 1.3.

1. $A \subseteq [\omega]^\omega$ *is called an* $\aleph_1$-Borel mad family *if* $A = \bigcup_{\alpha<\aleph_1} A_\alpha$*, where each* A_α *is a Borel a.d. family and* A *is a mad family.*

2. *Let $\mathbb{P}$ be a forcing partial order. An $\aleph_1$-Borel mad family A is called $\mathbb{P}$-*indestructible *if in the generic extension $V[G]$ by $\mathbb{P}$, $A^{V[G]} := \bigcup_{\alpha<\aleph_1} A_\alpha^{V[G]}$ is a mad family.*

In fact, our families A will even be unions of perfect sets A_α, so they could be called $\aleph_1$*-perfect mad families.*

Notice that in Definition 1.3, we could replace $\aleph_1$ by an arbitrary cardinal κ and define κ-Borel mad families analogously. A closer look at this allows us to isolate a new cardinal invariant.

DEFINITION 1.4. *Let $\mathfrak{a}_B$* (the Borel almost-disjointness number) *be the least infinite cardinal κ such that there exists a sequence $\{A_\alpha \mid \alpha < \kappa\}$ of Borel a.d. sets whose union is a mad family.*

It is clear that $\omega < \mathfrak{a}_B \leq \mathfrak{a}$, and also that if $\mathfrak{a}_B > \aleph_1$ then there are no $\mathbf{\Pi}^1_1$ mad families (since a $\mathbf{\Pi}^1_1$ set is a union of $\aleph_1$ Borel sets). Also, the following is an unpublished result of Dilip Raghavan (private communication), where $\mathfrak{t}$ stands for the *tower number*, the least size of a non-extendible tower in $\mathscr{P}(\omega)/\text{fin}$.

THEOREM 1.5 (Raghavan). *$\mathfrak{t} \leq \mathfrak{a}_B$.*

As a consequence, if $\mathfrak{t} > \aleph_1$ then there are no $\mathbf{\Pi}^1_1$ mad families.

Using essentially the same proof as that of our Main Theorem 1 above, we obtain the following:

MAIN THEOREM 2. *$\mathfrak{a}_B < \mathfrak{b}$ is consistent.*

In the interest of clarity, we will first present the proof of this cardinal inequality. After that it will be an easy matter to modify the proof so that it yields Main Theorem 1, i.e., the consistency of $\mathfrak{b} > \aleph_1$ with the existence of a $\mathbf{\Pi}^1_1$ mad family. We will start with a model of CH (L if we want the $\mathbf{\Pi}^1_1$ result) and extend it by a κ-iteration of Hechler forcing, for any uncountable regular cardinal κ, producing a model where $\mathfrak{b} = \kappa$ while $\mathfrak{a}_B = \aleph_1$. In the proof, we use an essential feature of Hechler forcing—preservation of ω-splitting families—first established by Baumgartner and Dordal in [1]. This allows us to construct an $\aleph_1$-perfect mad family in the ground model which is preserved in the Hechler iteration.

In section 2, we give some preliminary definitions in preparation for our construction. Then in section 3, we prove a Main Lemma, which will be the central technical tool used in establishing the proofs of both theorems. At the end of the section we prove Main Theorem 2 and sketch the modifications necessary to obtain Main Theorem 1.

§2. Preparing for the construction. To begin with, we do some preparatory work and lay out the foundations necessary for the construction of the Hechler-indestructible mad family.

As the primary component, we will consider partitions of an infinite subset of ω into infinitely many infinite sets, each one indexed by a finite sequence $\sigma \in \omega^{<\omega}$. To be precise, let D be an infinite subset of ω, and let

$$P := \{P_\sigma \mid \sigma \in \omega^{<\omega}\}$$

be a disjoint partition of D into infinite sets. We will call D the *domain of* P, and the elements of each P_σ will be enumerated in order, denoted by $P_\sigma = \{p_\sigma(0), p_\sigma(1), p_\sigma(2), \dots\}$. For each $f \in \omega^\omega$, let

$$\Phi(f) := \{p_{f\restriction n}(f(n)) \mid n < \omega\}$$

and consider $A_P := \{\Phi(f) \mid f \in \omega^\omega\}$. Then A_P is an almost disjoint subfamily of D of size $2^{\aleph_0}$, which forms a perfect set in the natural topology on $[D]^\omega$ (Φ is a homeomorphism between ω^ω and A_P). Any model of set theory containing P interprets the perfect set A_P according to its own reals. Our final mad family will consist of $\aleph_1$-many such perfect a.d. sets A_{P^α}, where $\{P^\alpha \mid \alpha < \aleph_1\}$ will be a sequence of partitions.

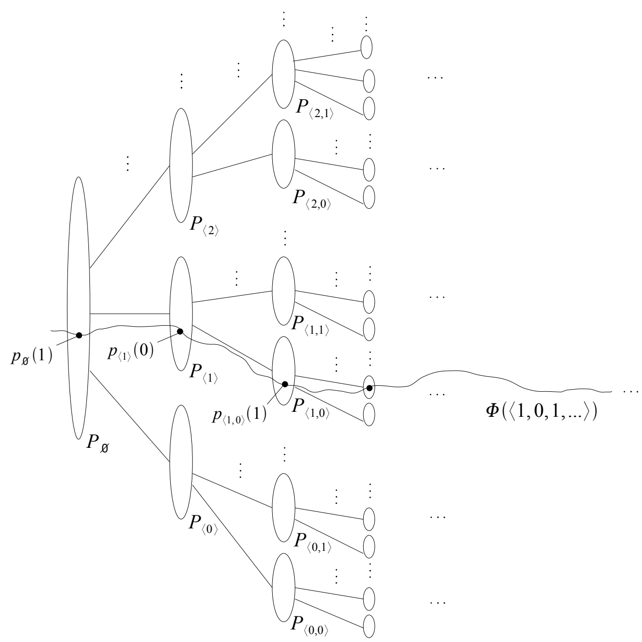

FIGURE 1. A partition P with domain D.

We also introduce the following additional notation. Let $\varphi : \omega^{<\omega} \setminus \{\varnothing\} \to [D]^{<\omega}$ be defined by

$$\varphi(\sigma) := \{p_{\sigma\restriction n}(\sigma(n)) \mid n < |\sigma|\},$$

and $\psi : \omega^{<\omega} \setminus \{\varnothing\} \to D$ be defined as follows: if $\sigma \in \omega^{<\omega} \setminus \{\varnothing\}$ and $n := |\sigma| - 1$ then

$$\psi(\sigma) := p_{\sigma\restriction n}(\sigma(n)).$$

Note that ψ is a bijection between $\omega^{<\omega} \setminus \{\varnothing\}$ and D, and that $\varphi(\sigma) = \{\psi(\sigma\restriction n) \mid 1 \le n \le |\sigma|\}$ and $\Phi(f) = \{\psi(f\restriction n) \mid 1 \le n < \omega\}$.

The rest of this section is a brief prelude to the actual construction. The idea is to start with a partition P contained in a countable model M, and then extend M using finite conditions (i.e., Cohen forcing) in order to add a new partition C (with some domain D_C). The new partition will give rise to a new perfect a.d. set A_C, in such a way that $A_P \cup A_C$ is still a.d. Later this consturction will be extended into the transfinite. The reason we want to use Cohen reals is that they satisfy the splitting property, which will be essential for proving the preservation of the mad family.

For a partition P and $\sigma \in \omega^{<\omega} \setminus \{\varnothing\}$, define

$$\tilde{I}(\sigma) := \{\psi(\tau) \mid \tau \subseteq \sigma \text{ or } \sigma \subseteq \tau\}.$$

Equivalently, we have $\tilde{I}(\sigma) := \bigcup \Phi''([\sigma])$. For $m \in D$, define $I(m) := \tilde{I}(\psi^{-1}(m))$. Each $I(m)$ is an infinite subset of D, and we let $\mathcal{I}_P$ be the ideal generated by all such $I(m)$, i.e., a set $X \subseteq \omega$ is defined to be in $\mathcal{I}_P$ if there are finitely many $m_0, \dots, m_k$ such that $X \subseteq^* \bigcup_{i=0}^{k} I(m_i)$. It is clear that $\mathcal{I}_P$ is a proper ideal, i.e., that D (and hence ω) is not in the ideal.

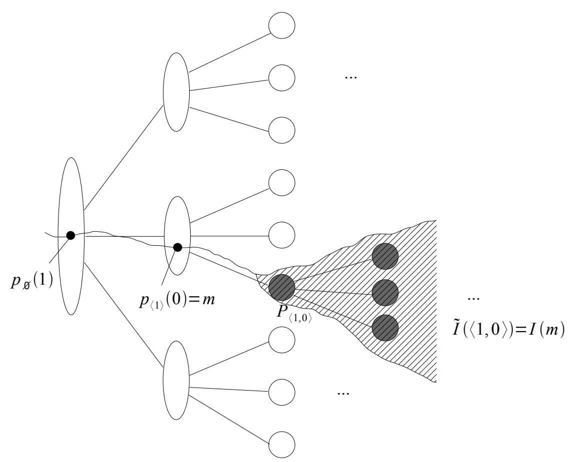

FIGURE 2. The ideal $\tilde{I}(\sigma) = I(m)$

Now, suppose M is a countable model of ZFC, and $P \in M$ is a partition of some D, as defined above. Using finite conditions, generically add a new partition $C = \{C_\sigma \mid \sigma \in \omega^{<\omega}\}$ of some domain D_C to M, in such a way that for all σ:

$$C_\sigma \cap \bigcup_{m < |\sigma|} I(m) = \varnothing.$$

To be precise, in M, define the partial order of functions $p : \omega^{<\omega} \times \omega \to 2$ with finite domain, ordered by extension, and additionally satisfying the following conditions:

1. for all σ, n, if $p(\sigma, n) = 1$ then $n \notin \bigcup_{m<|\sigma|} I(m)$, and
2. for all $\sigma \neq \tau$, there is no n such that $p(\sigma, n) = p(\tau, n) = 1$.

Once in the extension, define $C_\sigma := \{n \in \omega \mid (\bigcup G)(\sigma, n) = 1\}$, where G is generic over M. The domain D_C can be defined as $\bigcup\{C_\sigma \mid \sigma \in \omega^{<\omega}\}$. Using standard genericity arguments for Cohen forcing, plus the fact that the ideal $\mathcal{I}_P$ is proper, it is easy to see that $C := \{C_\sigma \mid \sigma \in \omega^{<\omega}\}$ forms a partition of D_C and satisfies the requirement.

We can now use C to define a new perfect a.d. family A_C. Let the functions $\Phi_C, \varphi_C, \psi_C$, the sets $I_C(m)$ and the ideal $\mathcal{I}_C$ be defined analogously using the new partition C. Moreover, let $\mathcal{I}_{P+C}$ denote the ideal generated by the $I(m)$'s as well as the $I_C(m)$'s, i.e., a set $X \subseteq \omega$ is in the ideal if there are $\ell_0, \dots, \ell_k$ and $m_0, \dots, m_r$ such that $X \subseteq^* I(\ell_0) \cup \dots \cup I(\ell_k) \cup I_C(m_0) \cup \dots \cup I_C(m_r)$.

We claim the following:

Lemma 2.1.

1. $A_P \cup A_C$ is almost disjoint, and

2. $\mathcal{I}_{P+C}$ is a proper ideal on $D \cup D_C$.

Proof. 1. Let $f, h \in \omega^\omega$, and we must show that $\Phi(f) \cap \Phi_C(h)$ is finite. Let m be any member of $\Phi(f)$, and let $\sigma \subseteq h$ be sufficiently long so that $m < |\sigma|$. Then for all τ with $\sigma \subseteq \tau \subseteq h$, C_τ is disjoint from $I(m)$, hence $\Phi_C(h) \cap I(m)$ is finite. On the other hand, $\Phi(f)$ is almost contained in $I(m)$. Therefore $\Phi(f) \cap \Phi_C(h)$ is finite.

2. Consider a finite union $I(\ell_0) \cup \dots \cup I(\ell_k) \cup I_C(m_0) \cup \dots \cup I_C(m_r)$. By standard genericity arguments, the Cohen real $C_\varnothing$ splits all reals in M. Since the $I(\ell_i)$'s are defined from P, it is clear that $\omega \setminus I(\ell_0) \cup \dots \cup I(\ell_k) \in M$. Therefore, there are infinitely many numbers in $C_\varnothing \setminus I(\ell_0) \cup \dots \cup I(\ell_k)$, and from these, only finitely many can be in $I_C(m_0) \cup \dots \cup I_C(m_r)$. ⊣

It is not hard to see that this way of extending A_P can also be used to extend any countable collection of a.d. families A_{P^i} contained in a countable model M—in that case, C_σ is defined so that it is disjoint from $\bigcup_{m,i<|\sigma|} I_i(m)$. This will be the main idea of our construction, but to guarantee that it is indestructible by an iteration of Hechler forcing, we must adjust and fine-tune this method.

§3. The Hechler-indestructible mad family. Recall the Hechler forcing partial order $\mathbb{D}$ consisting of conditions of the form $(s, f) \in \omega^{<\omega} \times \omega^\omega$ with $s \subseteq f$, ordered by $(s', f') \leq (s, f)$ iff $s \subseteq s'$ and $f' \geq f$. Hechler forcing satisfies the c.c.c., generically adds a dominating real, and moreover has many useful preservation properties. The one we will rely on in this proof is *preservation of ω-splitting families.* This property is typically used to show that the *splitting number* $\mathfrak{s}$ (the least size of a splitting family) remains small after a forcing iteration. Baumgartner and Dordal [1] proved it for Hechler forcing, and a similar result was proved by Dow in [5] about Laver forcing. We state it in a slightly stronger form (which is needed to show that the property is preserved by iterations, see also [2, Main Lemma 3.8]).

Fact 3.1 (Baumgartner and Dordal, [1]). *Let $\dot{a}$ be a $\mathbb{D}$-name for an element of $[\omega]^\omega$. Then there exist $\{a_i \mid i < \omega\}$, explicitly definable from the name $\dot{a}$, such* *that if c splits all a_i, then $\Vdash_{\mathbb{D}}$ "$\check{c}$ splits $\dot{a}$". This is still true in any iteration of $\mathbb{D}$ with finite support.*

This property of Hechler forcing can be used to show that if $X \subseteq [\omega]^\omega$ is an ω-splitting family, then it will remain so in the iterated Hechler extension. We will use it for a slightly different purpose, a kind of "properness"-version of preservation of ω-splitting families.

FACT 3.2. *Let M be a countable model and suppose $c \in [\omega]^\omega$ splits all reals in M. Then in the (iterated) Hechler extension $V[G]$, c splits all reals in $M[G]$.*

To see why this fact follows from the previous one, note that if $a \in M[G]$ then it has a name $\dot{a} \in M$, so the a_i from Fact 3.1 will also be in M because they are definable from $\dot{a}$. Therefore, c will split all a_i, and hence $\Vdash_{\mathbb{D}}$ "$\check{c}$ splits $\dot{a}$" (and the same is still true in an iteration of $\mathbb{D}$ with finite support).

We can now state our main result.

THEOREM 3.3 (Main Theorem 2). *Let V be a model of* CH, *and let G_κ be generic for the κ-iteration of Hechler forcing with finite support, for any uncountable regular cardinal κ. Then in $V[G_\kappa]$, $\mathfrak{b} = \kappa = 2^{\aleph_0}$ while $\mathfrak{a}_B = \aleph_1$.*

We will construct the $\aleph_1$-perfect mad family in V, by recursion on $\alpha < \aleph_1$ using ideas described in the previous section. Most of the effort will go into proving the main technical lemma concerned with the induction step, which we now state. The notation employed there is as follows: when P^i is a partition (of some domain D_i), then we denote by $\varphi_i, \Phi_i, \psi_i, I_i$ and A_i all the objects derived from it, and $\mathcal{I}_{\overline{P}}$ denotes the ideal generated by all the $I_i(m)$'s, for $i, m \in \omega$. For the new partition C, we denote by $\varphi_C, \Phi_C, \psi_C, I_C$ and A_C the objects derived from C, and $\mathcal{I}_{\overline{P}+C}$ denotes the ideal generated by the $I_i(m)$'s as well as the $I_C(m)$, for $i, m \in \omega$.

LEMMA 3.4 (Main Lemma). *Let M be a countable model of* ZFC *and let $\{P^i \mid i \in \omega\}$ be a sequence of partitions such that $P^i \in M$ for all i. Assume that $\forall i \neq j$ and $\forall f, g \in \omega^\omega$, $\Phi_i(f) \cap \Phi_j(g)$ is finite (i.e., $\bigcup_{i\in\omega} A_i$ is an a.d. family), and, moreover, assume that the ideal $\mathcal{I}_{\overline{P}}$ is proper (i.e., ω is not in the ideal).*

Then there exists a new partition $C := \{C_\sigma \mid \sigma \in \omega^{<\omega}\}$ of some domain D_C, lying outside M, which satisfies the following properties:

1. *For every $f, h \in \omega^\omega$ and every $i \in \omega$, $\Phi_i(f) \cap \Phi_C(h)$ is finite (hence, $\bigcup_{i\in\omega} A_i \cup A_C$ is a.d.).*
2. *The ideal $\mathcal{I}_{\overline{P}+C}$ is proper.*
3. *For every $Y \in M$, if Y is almost disjoint from $\Phi_i(f)$ for every $f \in \omega^\omega$ and every $i \in \omega$, then there exists an $h \in \omega^\omega$ such that $\Phi_C(h) \subseteq Y$.*

3*. *Suppose $V' \supseteq V$ is a model of set theory, $M' \supseteq M$ is a countable model in V', and every real in V which is splitting over M is still splitting over M'. Then for every $Y \in M'$, if Y is almost disjoint from $\Phi_i(f)$ for every $f \in \omega^\omega \cap V'$ and every $i \in \omega$, then there exists an $h \in \omega^\omega \cap V'$ such that $V' \models \Phi_C(h) \subseteq Y$ (i.e., condition 3 holds relativized to V' and M'.)*

In our application, V' will be $V[G_\kappa]$ for a κ-Hechler-generic G_κ, and M' will be $M[G_\kappa]$. In the proof of the lemma, conditions 3 and 3* are proved analogously, where the argument for 3* follows directly from the argument for 3. We will prove condition 3 in detail and then explain why it relativizes to V' and M' (the details will then be left out). Most of the work will involve building a combinatorially involved construction so that condition 3 is satisfied.

Note that if we add C generically using finite conditions, as we did in the previous section, we can easily satisfy conditions 1 and 2. Moreover, recall that Cohen reals are *splitting* over the ground model. To be precise, in the construction of the previous section, each C_σ was a Cohen subset of $\omega \setminus \bigcup_{m,i<|\sigma|} I_i(m)$, and thus split every set $Y \in M$ which had infinite intersection with $\omega \setminus \bigcup_{m,i<|\sigma|} I_i(m)$. This provides an idea for proving condition 3: if we could guarantee that the set $Y \in M$ has infinite intersection with $\omega \setminus \bigcup_{m,i<|\sigma|} I_i(m)$ for arbitrarily large σ, then we could inductively pick $h(n)$ so as to satisfy $\Phi_C(h) \subseteq Y$, using the splitting property of every $C_{h\restriction n}$. However, in general, this is clearly impossible: if $Y \subseteq I_i(m)$ for some m, then eventually σ will be long enough so that $i, m < |\sigma|$, and then Y will not have infinite intersection with the required set and we will not be able to use the splitting property of C_σ.

So we need to modify the construction, but the problem is that we must still guarantee condition 1. This turns out to be more tricky than may seem at first glance. To appreciate the difficulty, suppose $Y \in M$ is an infinite set which is "very close" to some $\Phi_i(f)$ but still almost disjoint from it, e.g., $Y = \{p^i_{f\restriction n}(f(n)+1) \mid n \in \omega\}$. If condition 1 of the lemma is to be satisfied, then $\Phi_C(h)$ must be almost disjoint from $\Phi_i(f)$ for any h. But how can this be achieved without the side-effect that $\Phi_C(h)$ is also almost disjoint from Y, making condition 3 impossible to satisfy? To alleviate this apparent tension between conditions 1 and 3 we will use a careful construction, such that, as σ grows longer, C_σ is disjoint from more and more sets of the form $I_i(m)$, but a special selection of those is excluded from this process. This will allow us, inductively, to select h so that every $C_{h\restriction n}$ splits Y, while making sure that condition 1 is not violated.

Proof of main lemma. The proof will proceed in three stages. First, we will give an recursive definition of the set C as we wish to have it, then we show that such a C actually exists, and finally we show that it satisfies conditions 1–3* of the lemma.

Part 1. The sets C_σ are defined by recursion on the length of σ. Moreover, to each σ and each $j < |\sigma|$, we associate another sequence $\tau_j(\sigma)$, called a *marker*, satisfying $|\tau_j(\sigma)| = |\sigma|$ and $\sigma \subseteq \sigma' \implies \tau_j(\sigma) \subseteq \tau_j(\sigma')$. This gives rise to the limit function $\bar{\tau}_j : \omega^\omega \to \omega^\omega$ defined by $\bar{\tau}_j(f) := \bigcup_{n>j} \tau_j(f\restriction n)$. The following convention will be useful here: variables $i^0, i^1, \ldots$ denote the digits of σ whereas $i^0_j, i^1_j, \ldots$ denote those of the markers $\tau_j(\sigma)$, and similarly for f versus $\bar{\tau}_j(f)$.

The variable Θ will be used to denote elements of $(\omega^{<\omega})^{<\omega}$, where for all $i < |\Theta|$, $|\Theta(i)| = |\Theta|$.

In what follows, we first explicitly describe the steps $k = 0$, $k = 1$ and $k = 2$ of the recursive construction, and then the step for arbitrary k. The first two steps are both necessary for the recursion, because the concept *forbidden value* only appears at stage $k \geq 1$. The "$k = 2$"-step is technically superfluous, and readers may skip directly to the "any k"-step; we have included it as an additional aid to help readers find their way through our construction.

- ($k = 0$)
  - $C_\varnothing$ is the disjoint union of infinitely many infinite sets $C_\varnothing^{\langle\langle i_0^0\rangle\rangle}$, where i_0^0 is an arbitrary integer. Each $C_\varnothing^{\langle\langle i_0^0\rangle\rangle}$ splits every $Y \in M$.
  - Require that for every i_0^0, $\psi_0(\langle i_0^0\rangle) \notin C_\varnothing^{\langle\langle i_0^0\rangle\rangle}$ (this is called an *excluded point*).
  - For every $i^0 \in \omega$, define

$$\tau_0(\langle i^0\rangle) := \langle i_0^0\rangle \;:\Longleftrightarrow\; c_\varnothing(i^0) \in C_\varnothing^{\langle\langle i_0^0\rangle\rangle}$$

    (here $c_\varnothing(i^0)$ refers to the enumeration of $C_\varnothing$ in increasing order, as defined in Section 2).
- ($k = 1$) The markers $\tau_0(\langle i^0\rangle) = \langle i_0^0\rangle$ have already been defined in the previous step.
  - Let $\sigma := \langle i^0\rangle$. Call i_0^1 a *forbidden value* if $\psi_0(\langle i_0^0, i_0^1\rangle) = \psi_C(\langle i^0\rangle) = c_\varnothing(i^0)$.
  - $C_{\langle i^0\rangle}$ is the disjoint union of infinitely many infinite sets $C_{\langle i^0\rangle}^{\langle\langle i_0^0,i_0^1\rangle,\langle i_1^0,i_1^1\rangle\rangle}$, where $\tau_0(\langle i^0\rangle) = \langle i_0^0\rangle$, i_0^1 is any natural number which is not a forbidden value, and i_1^0, i_1^1 are arbitrary. Define

$$X_\sigma := \bigcup\{I_j(\ell) \mid j, \ell < 1 \text{ and } \psi_j^{-1}(\ell) \perp \langle i_j^0\rangle\}.$$

    Each $C_{\langle i^0\rangle}^{\langle\langle i_0^0,i_0^1\rangle,\langle i_1^0,i_1^1\rangle\rangle}$ is disjoint from X_σ and splits every $Y \in M$ which has infinite intersection with $\omega \setminus X_\sigma$.
  - Require that $\psi_0(\langle i_0^0\rangle), \psi_0(\langle i_0^0, i_0^1\rangle), \psi_1(\langle i_1^0\rangle), \psi_1(\langle i_1^0, i_1^1\rangle) \notin C_{\langle i^0\rangle}^{\langle\langle i_0^0,i_0^1\rangle,\langle i_1^0,i_1^1\rangle\rangle}$, or, in other words, $\varphi_j(\langle i_j^0, i_j^1\rangle) \cap C_{\langle i^0\rangle}^{\langle\langle i_0^0,i_0^1\rangle,\langle i_1^0,i_1^1\rangle\rangle} = \varnothing$, for $j = 0, 1$ (*excluded points*).
  - For every $i^1 \in \omega$, define

$$\left.\begin{array}{c}\tau_0(\langle i^0, i^1\rangle) := \langle i_0^0, i_0^1\rangle \\ \text{and} \\ \tau_1(\langle i^0, i^1\rangle) := \langle i_1^0, i_1^1\rangle\end{array}\right\} :\Longleftrightarrow\; c_{\langle i^0\rangle}(i^1) \in C_{\langle i^0\rangle}^{\langle\langle i_0^0,i_0^1\rangle,\langle i_1^0,i_1^1\rangle\rangle}.$$

- ($k = 2$) The markers $\tau_0(\langle i^0, i^1\rangle) = \langle i_0^0, i_0^1\rangle$ and $\tau_1(\langle i^0, i^1\rangle = \langle i_1^0, i_1^1\rangle)$ have already been defined in the first two steps.
  - Let $\sigma := \langle i^0, i^1\rangle$. Call i_0^2 a *forbidden value* if $\psi_0(\langle i_0^0, i_0^1, i_0^2\rangle) \in \{\psi_C(\langle i^0\rangle), \psi_C(\langle i^0, i^1\rangle)\}$ and call i_1^2 a *forbidden value* if $\psi_1(\langle i_1^0, i_1^1, i_1^2\rangle) = \psi_C(\langle i^0, i^1\rangle)$.
  - $C_{\langle i^0,i^1\rangle}$ is the disjoint union of infinitely many infinite sets

$$C_{\langle i^0,i^1\rangle}^{\langle\langle i_0^0,i_0^1,i_0^2\rangle,\langle i_1^0,i_1^1,i_1^2\rangle,\langle i_2^0,i_2^1,i_2^2\rangle\rangle}$$

where $\tau_0(\langle i^0, i^1\rangle) = \langle i_0^0, i_0^1\rangle$, $\tau_1(\langle i^0, i^1\rangle) = \langle i_1^0, i_1^1\rangle$, i_0^2 and i_1^2 are any natural numbers which are not forbidden values, and i_2^0, i_2^1, i_2^2 are arbitrary. Define

$$X_\sigma := \bigcup\{I_j(\ell) \mid j, \ell < 2 \text{ and } \psi_j^{-1}(\ell) \perp \langle i_j^0, i_j^1\rangle\}.$$

Each $C_{\langle i^0,i^1\rangle}^{\langle\langle i_0^0,i_0^1,i_0^2\rangle,\langle i_1^0,i_1^1,i_1^2\rangle,\langle i_2^0,i_2^1,i_2^2\rangle\rangle}$ is disjoint from X_σ and splits every $Y \in M$ which has infinite intersection with $\omega \setminus X_\sigma$.

– Require that $\varphi_j(\langle i_j^0, i_j^1, i_j^2\rangle) \cap C_{\langle i^0,i^1\rangle}^{\langle\langle i_0^0,i_0^1,i_0^2\rangle,\langle i_1^0,i_1^1,i_1^2\rangle,\langle i_2^0,i_2^1,i_2^2\rangle\rangle} = \varnothing$, for $j = 0, 1, 2$ (*excluded points*).
– For every $i^2 \in \omega$, define

$$\left.\begin{array}{c} \tau_0(\langle i^0, i^1, i^2\rangle) := \langle i_0^0, i_0^1, i_0^2\rangle \\ \text{and} \\ \tau_1(\langle i^0, i^1, i^2\rangle) := \langle i_1^0, i_1^1, i_1^2\rangle \\ \text{and} \\ \tau_2(\langle i^0, i^1, i^2\rangle) := \langle i_2^0, i_2^1, i_2^2\rangle \end{array}\right\} :\Longleftrightarrow \begin{array}{l} c_{\langle i^0,i^1\rangle}(i^2) \in \\ \\ C_{\langle i^0,i^1\rangle}^{\langle\langle i_0^0,i_0^1,i_0^2\rangle,\langle i_1^0,i_1^1,i_1^2\rangle,\langle i_2^0,i_2^1,i_2^2\rangle\rangle}. \end{array}$$

This construction is continued in a similar fashion. The general inductive step looks as follows:

- (any k) The markers $\tau_j(\sigma)$ have been defined for all σ with $|\sigma| = k$ and $j < k$.
  – Fix σ such that $|\sigma| = k$. For $j < k$, call i_j^k a *forbidden value* if $\psi_j(\tau_j(\sigma)^\frown \langle i_j^k\rangle) \in \{\psi_C(\sigma{\restriction}m) \mid j < m \leq k\}$.
  – C_σ is the disjoint union of infinitely many sets C_σ^Θ, where

$$\Theta := \langle \tau_0(\sigma)^\frown \langle i_0^k\rangle, \dots, \tau_{k-1}(\sigma)^\frown \langle i_{k-1}^k\rangle, \langle i_k^0, \dots, i_k^k\rangle\rangle,$$

  with i_j^k not forbidden and $i_k^0, \dots, i_k^k$ arbitrary. Define

$$X_\sigma := \bigcup\{I_j(\ell) \mid j, \ell < k \text{ and } \psi_j^{-1}(\ell) \perp \tau_j(\sigma)\}.$$

  Each C_σ^Θ is disjoint from X_σ and splits all $Y \in M$ which have infinite intersection with $\omega \setminus X_\sigma$.
  – Require that $\varphi_j(\tau_j(\sigma)^\frown \langle i_j^k\rangle) \cap C_\sigma^\Theta = \varnothing$ for all $j < k$ and $\varphi_k(\langle i_k^0, \dots, i_k^k\rangle) \cap C_\sigma^\Theta = \varnothing$ (*excluded points*).
  – For every i^k, define

$$\left.\begin{array}{c} \forall j < k \ [\tau_j(\sigma^\frown \langle i^k\rangle) := \tau_j(\sigma)^\frown \langle i_j^k\rangle] \\ \text{and} \\ \tau_k(\sigma^\frown \langle i^k\rangle) := \langle i_0^k, \dots, i_k^k\rangle \end{array}\right\} :\Longleftrightarrow c_\sigma(i^k) \in C_\sigma^\Theta.$$

Thus we have completed the inductive definition of C.

Part 2. We show that the construction of C, as described above, can indeed be carried out. Since this time we want that not only every C_σ is splitting, but also every C_σ^Θ considered in the construction, we must add finite conditions in such a way that each C_σ^Θ is essentially a Cohen real over M. At first glance, a potential difficulty seems to arise from the fact that, in order to construct C_σ, we need to know the values of $\tau_j(\sigma')$ for $\sigma' \subseteq \sigma$, and these values are only known when the Cohen real has been added. However, we can avoid this difficulty by first adding

C^Θ_σ for *all possible* combinations of σ and Θ, each one being a Cohen subset of the relevant set. Afterwards, we can prune the tree to remove many of the C^Θ_σ's and leave only the ones that correspond to the construction described above.

To be precise, consider partial functions p with $\mathrm{dom}(p)$ being a finite subset of

$$\{(\sigma,\Theta,n)\in\omega^{<\omega}\times\left(\omega^{<\omega}\right)^{<\omega}\times\omega \mid |\Theta|=|\sigma|+1\}$$

and $\mathrm{ran}(p)=2$, ordered by extension, and satisfying the following conditions:

1. For every σ,Θ and n, if $p(\sigma,\Theta,n)=1$ then $n\notin X_{\sigma,\Theta}$, where
$$X_{\sigma,\Theta}:=\bigcup\{I_j(m)\mid j,m<|\sigma| \text{ and } \psi_j^{-1}(\ell)\perp(\mathrm{pr}_j(\Theta)\restriction|\Theta|-1)\},\text{ and}$$
2. for all $\sigma,\sigma',\Theta,\Theta'$ such that $(\sigma,\Theta)\neq(\sigma',\Theta')$, there is no n such that $p(\sigma,\Theta,n)=p(\sigma',\Theta',n)=1$.

Let G be the M-generic filter for this partial order, and in $M[G]$ define $C^\Theta_\sigma:=\{n\in\omega\mid(\bigcup G)(\sigma,\Theta,n)=1\}$. Genericity arguments for Cohen forcing show that all the C^Θ_σ are pairwise disjoint, and that every C^Θ_σ is disjoint from $X_{\sigma,\Theta}$ and splits every $Y\in M$ which has infinite intersection with $\omega\setminus X_{\sigma,\Theta}$.

Now, by induction on the length of σ, we can prune the tree given by the C^Θ_σ's and define the markers $\tau_j(\sigma)$ and the *forbidden values* accordingly. To be more precise, let σ be of length k and suppose that $C_{\sigma\restriction j}$ is already known for $j<k$. Since the values of $\tau_j(\sigma)$ for $j<k$ are then also known, we can compute the *forbidden values* at this step. Then, we throw away all C^Θ_σ except those where Θ is compatible with the already determined sequence of markers $\tau_j(\sigma)$ and the *forbidden values*, i.e., we keep only those C^Θ_σ where $\mathrm{pr}_j(\Theta)$ is of the form $\tau_j(\sigma)^\frown\langle i^k_j\rangle$ and i^k_j is not forbidden. After that, we still need to remove the *excluded points* from each relevant C^Θ_σ. Since this only requires changing finitely many elements, it does not affect the property of C^Θ_σ being a Cohen real.

Now C_σ can be defined as the union of the C^Θ_σ that we left behind and removed *excluded points* from. This allows us to extend τ_j and continue pruning the next levels. It is clear that in this manner we can achieve precisely the construction described above.

Part 3. Finally we show that C satisfies conditions 1–3 of the Lemma.

1. Let $f,h\in\omega^\omega$ and $j\in\omega$ be fixed. We must prove that $\Phi_j(f)\cap\Phi_C(h)$ is finite. There are now two methods for proving this. If f does not happen to be $\bar\tau_j(h)$, we can use an argument similar to Lemma 2.1. Otherwise, we will rely on the *excluded points* and the *forbidden values*.

- *Case 1:* $f\neq\bar\tau_j(h)$. Let $\sigma\subseteq h$ and $\tau\subseteq f$ be long enough so that $\tau_j(\sigma)\perp\tau$. Let $\ell:=\psi_j(\tau)$. Then clearly $\Phi_j(f)\subseteq I_j(\ell)$. Moreover, for any σ' such that $\sigma\subseteq\sigma'\subseteq h$ and $|\sigma'|>j,\ell$, we know that $\tau=\psi_j^{-1}(\ell)\perp\tau_j(\sigma')$, so, by construction, we know that $C^\Theta_{\sigma'}\cap I_j(\ell)=\varnothing$. This implies that $\Phi_C(h)\cap\Phi_j(f)$ is at most finite.
- *Case 2:* $f=\bar\tau_j(h)$. Ignore the first j values of $\Phi_j(f)$, and let $\sigma:=h\restriction(k+1)$, for $k>j$. Let $\tau:=f\restriction(m+1)$ for any m. Clearly, it is sufficient to show that $\psi_C(\sigma)\neq\psi_j(\tau)$.

– *Case* $(a) : m \leq k$. Then at stage k of the construction, $\psi_j(\tau)$ is an *excluded point* of $C_{\sigma\restriction k}^{\langle\dots,\tau_j(\sigma),\dots\rangle}$. But $\psi_C(\sigma) \in C_{\sigma\restriction k}^{\langle\dots,\tau_j(\sigma),\dots\rangle}$, so indeed $\psi_C(\sigma) \neq \psi_j(\tau)$.
– *Case* $(b) : k < m$. Let $i_j^m := \tau(m)$. Then at stage m of the construction, i_j^m cannot be a *forbidden value*. Then, by definition, $\psi_j(\tau) \neq \psi_C(h\restriction r)$ for any r with $j < r \leq m$, in particular for $r = k+1$. Therefore $\psi_j(\tau) \neq \psi_C(\sigma)$.

2. To show that $\mathcal{I}_{\overline{P}+C}$ is proper, consider any finite union $I_{j_0}(\ell_0) \cup \dots \cup I_{j_k}(\ell_k) \cup I_C(m_0) \cup \dots \cup I_C(m_r)$. Let $Z := I_{j_0}(\ell_0) \cup \dots \cup I_{j_k}(\ell_k)$, and note that Z is in M, since it is constructed from partitions contained in M. Recall that, by construction, every $C_\varnothing^\Theta$ splits every real in M, so in particular, it splits $\omega \setminus Z$. Therefore, there are infinitely many elements in $C_\varnothing^\Theta \setminus Z$. From those, only finitely many can be in $I_C(m_0) \cup \dots \cup I_C(m_r)$. Hence, infinitely many elements are not in $I_{j_0}(\ell_0) \cup \dots \cup I_{j_k}(\ell_k) \cup I_C(m_0) \cup \dots \cup I_C(m_r)$.

3. This is the essence of the proof, and the main reason for setting up the construction as we have done it. Suppose Y is an infinite subset of ω in M, and $Y \cap \Phi_j(g)$ is finite for all j and all $g \in \omega^\omega$. The goal is to construct an h such that $\Phi_C(h) \subseteq Y$.

First, we build functions $g_j \in \omega^\omega$ in M, and a sequence $Y \supseteq Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \dots$ of infinite sets in M, making sure that for every j the following condition is satisfied:

$$(*)_j : \quad Y_j \cap \tilde{I}_j(\rho) \text{ is finite for every } \rho \text{ incompatible with } g_j.$$

Start by constructing $g_0 := \langle i_0^0, i_0^1, i_0^2, \dots \rangle$, taking care of the partition P^0. Consider two cases: (a) there exists an i_0^0 such that $Y \cap \tilde{I}_0(\langle i_0^0 \rangle)$ is infinite, and (b) $Y \cap \tilde{I}_0(\langle i_0^0 \rangle)$ is finite for any choice of i_0^0. If case (b) occurs, pick all the $i_0^0, i_0^1, \dots$ arbitrarily (they are irrelevant), and set $K_0 := 0$ (this encodes the fact that case (b) occurred at the 0-th step). If case (a) occurs, fix this i_0^0 and continue. Consider two cases: (a) there exists an i_0^1 such that $Y \cap \tilde{I}_0(\langle i_0^0, i_0^1 \rangle)$ is infinite, and (b) $Y \cap \tilde{I}_0(\langle i_0^0, i_0^1 \rangle)$ is finite for any choice of i_0^1. If case (b) occurs, pick all the remaining $i_0^1, i_0^2, \dots$ arbitrarily, and set $K_0 := 1$; if case (a) occurs, fix this i_0^1 and continue. Go on in a similar fashion: if $g_0\restriction k$ is defined, there are two cases: (a) there exists an i_0^k such that $Y \cap \tilde{I}_0(g_0\restriction k^\frown\langle i_0^k \rangle)$ is infinite, and (b) $Y \cap \tilde{I}_0(g_0\restriction k^\frown\langle i_0^k \rangle)$ is finite for any choice of i_0^k. In case (b) pick $i_0^k, i_0^{k+1}, \dots$ arbitrarily and set $K_0 := k$; in case (a) fix i_0^k and continue.

This way we define $g_0 = \langle i_0^0, i_0^1, i_0^2, \dots \rangle$. If case (b) occurred at any finite stage k, we also have $K_0 := k$, otherwise K_0 is undefined. Now, we want to shrink Y to a new infinite set Y_0, in such a way that condition $(*)_0$ is satisfied. There are two possibilities.

(i) If case (b) occurred at some stage, then let $Y_0 := Y \cap \tilde{I}_0(g_0\restriction K_0)$ (or $Y_0 = Y$ if $K_0 = 0$). By construction, Y_0 is infinite, and it is not hard to check that Y_0 has finite intersection with $\tilde{I}_0(\rho)$ whenever ρ is incompatible with g_0 (in fact, this holds for all ρ except $\rho \subseteq g_0\restriction K_0$). Therefore, condition $(*)_0$ is satisfied by Y_0.

(ii) If case (a) occurred throughout the construction, then notice the following: for every n, there is a $y \in Y$ such that $g_0 \restriction n \subseteq \psi_0^{-1}(y)$. So, for each n, pick one such y_n, and let $Y_0 := \{y_0, y_1, y_2, \dots\}$. Clearly Y_0 is an infinite subset of Y. Moreover, if ρ is any sequence incompatible with g_0, then, letting n be least such that $\rho(n) \neq g_0(n)$, we see that $\tilde{I}_0(\rho)$ can contain at most n members of Y (because for any y_m for $m > n$ we have $g_0 \restriction m \subseteq \psi_0^{-1}(y_m)$ and hence $y_m \notin \tilde{I}_0(\rho)$). Therefore condition $(*)_0$ is satisfied by Y_0.

Note that, in either case, Y_0 is explicitly constructed using information encoded in the partition $P_0 \in M$, so Y_0 is also in M.

Now we continue with the construction of $g_1 := \langle i_1^0, i_1^1, i_1^2, \dots \rangle$ using Y_0 instead of Y, taking care of the partition P^1 instead of P^0. Consider two cases: (a) there exists an i_1^0 such that $Y_0 \cap \tilde{I}_1(\langle i_0^0 \rangle)$ is infinite, and (b) $Y_0 \cap \tilde{I}_1(\langle i_1^0 \rangle)$ is finite for any choice of i_1^0. If case (b) occurs, pick all the $i_1^0, i_1^1, \dots$ arbitrarily and set $K_1 := 0$; if case (a) occurs, fix this i_1^0 and continue, *etc.* After we have defined g_1, let Y_1 be an infinite subset of Y_0, constructed in the same way as Y_0 was constructed out of Y, i.e., so that condition $(*)_1$ is satisfied, and again $Y_1 \in M$.

It is clear that this method can be continued, so at each step j we deal with the partition P^j, define $g_j = \langle i_j^0, i_j^1, i_k^2, \dots \rangle$ and an infinite set $Y_j \in M$, following the same procedure, and make sure that condition $(*)_j$ is satisfied.

Now, we can define the function $h := \langle i^0, i^1, i^2, \dots \rangle$ so that the following three conditions are satisfied for every k:

1. $c_{\langle i^0,\dots,i^{k-1}\rangle}(i^k) \in C_{\langle i^0,\dots,i^{k-1}\rangle}^{\langle \langle i_0^0,\dots,i_0^k \rangle,\dots,\langle i_k^0,\dots,i_k^k \rangle \rangle}$,
2. $c_{\langle i^0,\dots,i^{k-1}\rangle}(i^k) \in Y_k$, and
3. $c_{\langle i^0,\dots,i^{k-1}\rangle}(i^k) \notin \bigcup_{j \leq k} \Phi_j(g_j)$.

The first condition inductively guarantees that for every k and every $j < k$, $\tau_j(\langle i^0, \dots, i^k \rangle) = \langle i_j^0, \dots, i_j^k \rangle$. The third condition is crucial: it is to ensure that we will not run into trouble with *forbidden values* i_j^m for $j \leq k < m$ in the future. This is the only place in the argument where the assumption that Y is a.d. from all $\Phi_j(g)$ is needed.

To see that the numbers i^k satisfying conditions 1–3 can indeed be chosen, proceed inductively. Suppose $\langle i^0, \dots, i^{k-1} \rangle$ has already been defined. Condition 3 inductively implies that i_j^k for $j < k$ are not *forbidden values*, therefore we can consider the set $C_{\langle i^0,\dots,i^{k-1}\rangle}^{\langle \langle i_0^0,\dots,i_0^k \rangle,\dots,\langle i_k^0,\dots,i_k^k \rangle \rangle}$. Recall that this set was defined so that it splits every $Y' \in M$ which has infinite intersection with $\omega \setminus X_{\langle i^0,\dots,i^{k-1}\rangle}$, where

$$X_{\langle i^0,\dots,i^{k-1}\rangle} = \bigcup \{ I_j(\ell) \mid j, \ell < k \text{ and } \psi_j^{-1}(\ell) \perp \langle i_j^0, \dots, i_j^{k-1} \rangle \}.$$

However, condition $(*)_k$ implies that Y_k is almost disjoint from any $I_j(\ell)$ with $\psi_j^{-1}(\ell) \perp g_j$. In particular, it is almost disjoint from any $I_j(\ell)$ with $\psi_j^{-1}(\ell) \perp \langle i_j^0, \dots, i_j^{k-1} \rangle$. But then Y_k must be almost disjoint from a finite union of such sets, and therefore, have infinite intersection with $\omega \setminus X_{\langle i^0,\dots,i^{k-1}\rangle}$.

Therefore $C_{\langle i^0,\dots,i^{k-1}\rangle}^{\langle\langle i_0^0,\dots,i_0^k\rangle,\dots,\langle i_k^0,\dots,i_k^k\rangle\rangle}$ splits Y_k, so there are infinitely many numbers n in the set $Y_k \cap C_{\langle i^0,\dots,i^{k-1}\rangle}^{\langle\langle i_0^0,\dots,i_0^k\rangle,\dots,\langle i_k^0,\dots,i_k^k\rangle\rangle}$. Now we recall the fact that, by assumption, $|Y_k \cap \Phi_j(g_j)| < \omega$ for every $j \le k$. Therefore, it is possible to pick an n even so that condition 3 (at induction step k) is also satisfied. So we pick such an n and let i^k be such that $C_{\langle i^0,\dots,i^{k-1}\rangle}(i^k) = n$. This completes the induction step.

We thus construct the entire function $h = \langle i^0, i^1, \dots \rangle$. The second condition implies that $\Phi_C(h) \subseteq Y$, as had to be shown.

3.* Let $V' \supseteq V$, $M' \supseteq M$, $M' \in V'$ and assume that every real in V which is splitting over M is still splitting over M'. Notice that a "splitting real" here can refer not just to an element of $[\omega]^\omega$ but also to an element of $[D]^\omega$, for some fixed infinite $D \subseteq \omega$, provided that $D \in M$.

Recall that the new partition C is constructed out of many reals of the form C_σ^Θ, where each C_σ^Θ is a splitting real over M in the sense of the space $[\omega \setminus X_\sigma]^\omega$, where

$$X_\sigma := \bigcup\{I_j(\ell) \mid j, \ell < |\sigma| \text{ and } \psi_j^{-1}(\ell) \perp \tau_j(\sigma)\},$$

and $\omega \setminus X_\sigma$ is in M. By assumption, the same real C_σ^Θ is still splitting over M', again in the sense of the space $[\omega \setminus X_\sigma]^\omega$ (interpreted in V').

In point 3 above, this splitting property of the C_σ^Θ over M was the main tool in defining the function h. Now, we simply reproduce the entire proof relativized to V' and M', but using splitting over M'. To be specific, if Y is a real in M', and if $|Y \cap \Phi_i(g)| < \omega$ for every $i \in \omega$ and every $g \in \omega^\omega$ in V', we construct functions g_j as before, except that now g_j may belong to M'. However, since the sets X_σ are the same and C_σ^Θ splits all reals in M' which have infinite intersection with $\omega \setminus X_\sigma$, we can apply the same argument as before to produce a function h in V', such that $V' \models \Phi_C(h) \subseteq Y$. This completes the proof of the Main Lemma. $\dashv$

Proof of main theorem 2. Let V be a model of CH, and κ an uncountable regular cardinal. We will construct a $\mathbb{D}_\kappa$-indestructible $\aleph_1$-perfect mad family in V. If A is such a family, we will denote by $A^{V[G_\kappa]}$ the re-interpreted $\aleph_1$-perfect mad family, i.e., the $\aleph_1$-union of the re-interpreted perfect sets. Before proceeding with the construction, we show that preservation in iterations of length $\aleph_1$ is sufficient.

Claim. *If an $\aleph_1$-perfect mad family A is $\mathbb{D}_{\aleph_1}$-indestructible, then it is also $\mathbb{D}_\kappa$-indestructible.*

Proof. For a countable set $S \subseteq \kappa$, let $\mathbb{D}_S$ denote the iteration of Hechler forcing with support S. It is known that the κ-iteration of Hechler forcing is the *direct limit* of iterations $\mathbb{D}_S$ where S ranges over countable subsets of κ. This is true because Hechler forcing is a Suslin c.c.c. forcing notion, see e.g. [3, p 54] for a proof. In particular, any new real added in the iteration $\mathbb{D}_\kappa$ is already added by some $\mathbb{D}_S$.

Let A be an $\aleph_1$-perfect mad family in V and suppose it is not $\mathbb{D}_\kappa$-indestructible. Then there is a $Y \in V[G_\kappa]$ which is almost disjoint from $A^{V[G_\kappa]}$. By the above, there is a countable $S \subseteq \kappa$ such that Y is in $V[G_S]$. Thus in $V[G_S]$, Y is almost disjoint from $A^{V[G_S]}$, and so $V[G_S] \models$ "A is not maximal". Since there is a canonical isomorphism between $\mathbb{D}_S$ and $\mathbb{D}_\gamma$ where $\gamma < \aleph_1$ is the order-type of S, it follows that also $V[G_\gamma] \models$ "A is not maximal". This proves that A is not $\mathbb{D}_{\aleph_1}$-indestructible. ⊣

Because of this claim, it suffices to construct a $\mathbb{D}_{\aleph_1}$-indestructible $\aleph_1$-perfect mad family in V.

Now we can proceed with the construction. First, note that in V there is a set $\{\dot{x}_\alpha \mid \alpha < \aleph_1\}$ of *canonical* $\mathbb{D}_{\aleph_1}$*-names for reals*, i.e., such that if $\dot{z}$ is any $\mathbb{D}_{\aleph_1}$-name for a real then $\Vdash_{\mathbb{D}_{\aleph_1}} \dot{z} = \dot{x}_\alpha$ for some α.

Next, we construct an increasing sequence of countable models

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq \ldots \qquad (\alpha < \aleph_1)$$

covering all the names $\{\dot{x}_\alpha \mid \alpha < \aleph_1\}$, and simulaneously construct a sequence $\{P^\alpha \mid \alpha < \aleph_1\}$ of partitions, recursively generated using the main lemma. The corresponding perfect a.d. families will be denoted by A_α, and the ideal generated by it by $\mathcal{I}_\alpha$. We also write $\mathcal{I}_{<\alpha}$ for the ideal generated by all $I_\beta(m)$ for $\beta < \alpha$. The induction will guarantee that, for each α, the following four conditions are satisfied:

(IH1) $\dot{x}_\beta \in M_\alpha$ for all $\beta < \alpha$
(IH2) $P^\beta \in M_\alpha$ for all $\beta < \alpha$,
(IH3) $\bigcup_{\beta<\alpha} A_\beta$ is an a.d. family, and
(IH4) the ideal $\mathcal{I}_{<\alpha}$ is proper.

At stage 0, we let M_0 be any countable model of enough of ZFC. Next, suppose M_α is a countable model, the P^β for $\beta < \alpha$ have already been constructed and all four inductive conditions are satisfied. Then we are in the right position to apply the main lemma. From it, we obtain a new partition, P^α, with conditions 1 and 2 of the main lemma making sure that IH3 and IH4 will be satisfied at the next step of the induction. Now, let $M_{\alpha+1}$ be any model containing M_α as a subset, as well as the name $\dot{x}_\alpha$ and the new partition P_α. This makes sure that IH1 and IH2 are also satisfied at step $\alpha + 1$. At limit steps $\lambda < \aleph_1$, let $M_\lambda := \bigcup_{\alpha<\lambda} M_\alpha$. It is clear that conditions IH1–IH4 are satisfied at this step.

Note that the construction of P^0 is a trivial application of the main lemma; since the ideal $\mathcal{I}_{<0}$ is empty, P^0 is simply a partition of Cohen reals over M_0, which split all sets $Y \in M_0$.

Let $A := \bigcup_{\alpha<\aleph_1} A_\alpha$. This is our Hechler-indestructible $\aleph_1$-perfect mad family. First, let us see that A is mad in V. Take any $Y \in [\omega]^\omega$, and note that since the sequence of models M_α covers all names for reals (modulo equivalence), in particular it covers ground model reals, so there is an M_α such that $Y \in M_\alpha$. By point 3 of the main lemma, either there is an $f \in \omega^\omega$ and a $\beta < \alpha$ such that Y has infinite intersection with $\Phi_\beta(f)$, or there is an $h \in \omega^\omega$ such that $\Phi_\alpha(h) \subseteq Y$, so in either case Y has infinite intersection with A.

Now, let us check that A is preserved in $V[G_{\aleph_1}]$, the $\mathbb{D}_{\aleph_1}$-extension of V. Take any $Y \in [\omega]^\omega \cap V[G_{\aleph_1}]$, and let $\dot{Y}$ be a name for Y. Without loss of generality we may assume that $\dot{Y}$ is a canonical $\mathbb{D}_{\aleph_1}$-name, hence there is an M_α such that $\dot{Y} \in M_\alpha$. As $\mathbb{D}_{\aleph_1}$ is Suslin c.c.c., $G_{\aleph_1}$ is generic over M_α. By Fact 3.2, every real in V which is splitting over M is splitting over $M[G_{\aleph_1}]$, so we can apply point 3^* of the main lemma with $V' = V[G_{\aleph_1}]$ and $M' = M[G_{\aleph_1}]$. Therefore, either there is an $f \in \omega^\omega \cap V[G_{\aleph_1}]$ and a $\beta < \alpha$ such that Y has infinite intersection with $\Phi_\beta(f)$, or there is an $h \in \omega^\omega \cap V[G_{\aleph_1}]$ such that $\Phi_\alpha(h) \subseteq Y$. In either case Y has infinite intersection with $A^{V[G_{\aleph_1}]}$.

Thus, the $\aleph_1$-perfect mad family A is preserved in the generic $\mathbb{D}_{\aleph_1}$-extension $V[G_{\aleph_1}]$, and therefore also in the generic $\mathbb{D}_\kappa$-extension, for all regular uncountable κ. This witnesses the fact that $\mathfrak{a}_B = \aleph_1$ in $V[G_\kappa]$. On the other hand, $\mathfrak{b} = \kappa$ in $V[G_\kappa]$, and this completes the proof. ⊣

So we have proved the consistency of $\mathfrak{a}_B < \mathfrak{b}$, and it remains only to verify that the proof can be adapted to yield the consistency of $\mathfrak{b} > \aleph_1 +$"there is a $\mathbf{\Pi}^1_1$ mad family". For this, we start with L instead of an arbitrary model of CH, and repeat the construction.

Theorem 3.5 (Main Theorem 1). *Let G_κ be generic for the κ-iteration of Hechler forcing with finite support, for any uncountable regular cardinal κ. Then in $L[G_\kappa]$, $\mathfrak{b} = \kappa = 2^{\aleph_0}$ while there exists a $\mathbf{\Pi}^1_1$ mad family.*

Proof. We modify the previous proof as follows: fix some uniform coding of partitions P^α in L by reals. In the induction step α of the proof, instead of picking some P^α given to us by the main lemma, as we did before, we pick the P^α with the $<_L$-least code. Moreover, instead of picking the countable models M_α arbitrarily, we choose the least L_{δ_α}, for a limit ordinal $\delta_\alpha < \aleph_1$, which contains all the relevant objects as elements.

Let $\mathcal{P}$ denote the set of all (codes of) $\{P^\alpha \mid \alpha < \aleph_1\}$ produced in this new proof. As always, the absoluteness of $<_L$ and everything else involved in this construction implies that the definition of the set $\mathcal{P}$ is absolute between L and a sufficiently large L_δ. Therefore, we may write $P \in \mathcal{P}$ iff $\exists L_\delta\ (P \in L_\delta \wedge L_\delta \models P \in \mathcal{P})$. By standard arguments, this implies that $\mathcal{P}$ is $\mathbf{\Sigma}^1_2$-definable.

For $P \in \mathcal{P}$, let A_P denote the perfect a.d. family based on P (i.e., the A_α for $P = P^\alpha$). By what we already proved, $A = \bigcup_{P \in \mathcal{P}} A_P$ is preserved in $L[G_\kappa]$. Then $A^{L[G_\kappa]}$ can be given by the following definition:

$$a \in A^{L[G_\kappa]} \iff \exists P\, (P \in \mathcal{P} \wedge a \in A_P^{L[G_\kappa]}),$$

which is a $\mathbf{\Sigma}^1_2$ statement. So in $L[G_\kappa]$ there is a $\mathbf{\Sigma}^1_2$ mad family, and by Törnquist's equivalence, also a $\mathbf{\Pi}^1_1$ mad family. Thus we have obtained a model of $\mathfrak{b} > \aleph_1$ + "there is a $\mathbf{\Pi}^1_1$ mad family". ⊣

§4. Open questions. Many questions about projective mad families are still open. For instance, we do not know how the existence of a $\mathbf{\Pi}^1_1$ mad family is related to other statements about projective regularity properties. Since the original proof of Mathias (Theorem 1.1) involves a Ramsey-style property, the following question is interesting:

QUESTION 4.1. *Does the statement "all $\mathbf{\Sigma}^1_2$ sets have the Ramsey property" imply that there is no $\mathbf{\Pi}^1_1$ mad family?*

On the other hand, our own proof relies heavily on splitting reals and the preservation of splitting families by Hechler forcing, so one may ask whether the following holds:

QUESTION 4.2. *Does "there is no $\mathbf{\Pi}^1_1$ mad family" imply that for all r, $L[r] \cap [\omega]^\omega$ is not a splitting family?*

The most interesting result in this context would be a characterization theorem.

QUESTION 4.3. *Is there some notion of transcendence over L which is equivalent to the statement "there is no $\mathbf{\Pi}^1_1$ mad family"?*

Other questions concern the cardinal invariant $\mathfrak{a}_B$. Although we have established the consistency of $\mathfrak{a}_B < \mathfrak{b}$, it is not clear whether the converse holds.

QUESTION 4.4. *Is $\mathfrak{a}_B \leq \mathfrak{b}$ provable in* ZFC*, or is $\mathfrak{b} < \mathfrak{a}_B$ consistent?*

Concerning lower bounds for $\mathfrak{a}_B$, the following is a conjecture of Dilip Raghavan (where $\mathfrak{h}$ stands for the *distributivity number*):

CONJECTURE 4.5. $\mathfrak{h} \leq \mathfrak{a}_B$.

Since the canonical method of increasing $\mathfrak{h}$ is to iterate Mathias forcing, this conjecture seems closely related to Question 4.1.

Concerning upper bounds for $\mathfrak{a}_B$, a question related to Question 4.2 would be whether the splitting number $\mathfrak{s}$ (the least size of a splitting family) is an upper bound for $\mathfrak{a}_B$. Since we have also used the countability of the α's in our construction, it may be more realistic to expect only the weaker result that if $\mathfrak{s} = \aleph_1$ then $\mathfrak{a}_B = \aleph_1$.

QUESTION 4.6. *Is $\mathfrak{a}_B \leq \mathfrak{s}$ provable in* ZFC*? Or, at least, is $\mathfrak{s} = \aleph_1 \to \mathfrak{a}_B = \aleph_1$ provable in* ZFC*?*

In a recent result, Raghavan and Shelah [14] showed the weaker statement that if $\mathfrak{d} = \aleph_1$ then $\mathfrak{a}_B = \aleph_1$.

Finally, recall that the a.d. families A_α we constructed were, in fact, closed sets. We can define the cardinal invariant $\mathfrak{a}_{\text{closed}}$ as the least number of *closed* a.d. sets whose union is a mad family. It is obvious that $\mathfrak{a}_B \leq \mathfrak{a}_{\text{closed}}$, and that our proof actually shows the stronger result $\text{Con}(\mathfrak{a}_{\text{closed}} < \mathfrak{b})$. However, it is not clear whether the two cardinal invariants are different.

QUESTION 4.7. *Is $\mathfrak{a}_B = \mathfrak{a}_{\text{closed}}$ provable in* ZFC*, or is $\mathfrak{a}_B < \mathfrak{a}_{\text{closed}}$ consistent?*

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