Truth and definability Lemma

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Contents

What's in a name

Let V our universe. Let $(\mathbb{P}, \leq, \mathbf{1})$ be a partial ordering known as the forcing relation. We define a **name** in the following way:

Definition

τ is a $\mathbb{P}-$name iff τ is a relation and

$$\forall(\sigma, p) \in \tau[\sigma \text{ is a } \mathbb{P}\text{-name} \wedge p \in \mathbb{P}]$$

We call $V^{\mathbb{P}}$ the class of all $\mathbb{P}$-names.

What's in a name

Let M be a (meta-)countable transitive model of ZFC and let $(\mathbb{P}, \leq, \mathbf{1}) \in M$, then $M^{\mathbb{P}}$ is the class of all $\mathbb{P}$-names in M i.e.

$$M^{\mathbb{P}} = V^{\mathbb{P}} \cap M$$

What's in a name

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Remark

From now on, $\mathbb{P}$ means $(\mathbb{P}, \leq, \mathbf{1})$, unless stated otherwise.

For what it's worth...

Let $G \subseteq \mathbb{P}$, then $val_G(\tau)$ is defined the following way,

Definition

If τ is a $\mathbb{P}$-name, then we define

$$val_G(\tau) = \tau_G = \{\sigma_G \mid \exists p \in G[(\sigma, p) \in \tau]\}$$

Now, we have a class

$$M[G] = \{\tau_G \mid \tau \in M^{\mathbb{P}}\}$$

For what it's worth...

$M[G]$ has lots of nice properties, whenever G is a filter or generic-filter!

Contents

The Force Awakens

Let $\mathbb{P} \in M$ be a forcing poset, then $\mathcal{FL}_{\mathbb{P}}$ be the set of all $\{\in\}-$formulas with free variables replaced by members of $V^{\mathbb{P}}$ and $\mathcal{AL}_{\mathbb{P}}$ be the set of all atomic $\{\in\}-$formulas with free variables replaced by members of $V^{\mathbb{P}}$.

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We now define a relation $\Vdash^*$ on $\mathbb{P} \times \mathcal{AL}_{\mathbb{P}}$.

definition

1. $p \Vdash^* \tau = \nu$ iff
   $\forall \sigma \in [dom(\tau) \cup dom(\nu)] \forall q \leq p (q \Vdash^* \sigma \in \tau \leftrightarrow q \Vdash^* \sigma \in \nu)$.
2. $p \Vdash^* \tau \in \nu$ iff
   $\{q \leq p \mid \exists(\sigma, r) \in \nu [q \leq r \wedge \sigma = \tau]\}$ is dense below p.

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Let $\tau_1, \tau_2, \sigma_1, \sigma_2 \in V^{\mathbb{P}}$ and $p_1, p_2 \in \mathbb{P}$, then

1. $(p_1, \sigma_1 \in \tau_1) R (p_2, \sigma_2 = \tau_2)$ iff
   $[\sigma_1 \in trcl(\sigma_2) \vee \sigma_1 \in trcl(\tau_2)]$ and $[\tau_1 = \sigma_2 \vee \tau_1 = \tau_2]$
2. $(p_1, \sigma_1 = \tau_1) R (p_2, \sigma_2 \in \tau_2)$ iff
   $\sigma_1 = \sigma_2$ and $\tau_1 \in trcl(\tau_2)$
3. $(p_1, \sigma_1 \in \tau_1) R (p_2, \sigma_2 \in \tau_2)$ and $(p_1, \sigma_1 = \tau_1) R (p_2, \sigma_2 = \tau_2)$ doesn't hold.

The Force Awakens

R is set-like from the previous definition and to see that it is well-founded, we are going to define a function $F : \mathbb{P} \times \mathcal{AL}_{\mathbb{P}} \to Ord$ s.t. aRb then $F(a) < F(b)$.

definition

$\rho((p_1, \sigma = \tau)) = \rho((p_2, \sigma \in \tau)) = max(rank(\sigma), rank(\tau))$

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definition

$\rho((p_1, \sigma = \tau)) = \rho((p_2, \sigma \in \tau)) = max(rank(\sigma), rank(\tau))$

But this doesn't do the trick, since

$$(p, \phi_1)R(p', \phi_2) \to \rho((p, \phi_1)) \leq \rho((p', \phi_2))$$

Notice the following thing,

The Force Awakens

definition

Let $ty : \mathcal{AL}_{\mathbb{P}} \to \{0, 1, 2\}$ by the following way:

1. $ty(\tau = \sigma) = 1$
2. $ty(\tau \in \sigma) = 0$ if $rank(\tau) < rank(\sigma)$
3. $ty(\tau \in \sigma) = 2$ if $rank(\tau) \geq rank(\sigma)$

Then, the following function $F(p, \phi) = 3 \times \rho(\phi) + ty(\phi)$ actually satisfies the needed property.

Now, we can extend the definition further to $\mathbb{P} \times \mathcal{FL}_{\mathbb{P}}$

definition

1. If $\phi = \psi \wedge \theta$, $p \Vdash^* \psi \wedge \theta$ iff $p \Vdash^* \psi$ and $p \Vdash^* \theta$
2. If $\phi = \neg\psi$, $p \Vdash^* \neg\psi$ iff $\neg\exists q[q \leq p \wedge q \Vdash^* \psi]$
3. If $\phi = \exists x\psi(x)$, iff $\{q \leq p \mid \exists\tau \in V^{\mathbb{P}} q \Vdash^* \psi(\tau)\}$ is dense below p

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All the other connectives can be written in the form of these three.

Jargon

Some still might find $\Vdash^*$ a little bit strange, but it has some "nice" properties. Let $p \in \mathbb{P}$,

- $p \Vdash^* \tau = \tau$.
- For any $(x, r) \in \tau$ with $r \geq p$, $p \Vdash^* x \in \tau$.
- If $p \Vdash^* \phi$ then $\forall q \leq p \; q \Vdash^* \phi$
- If $p \Vdash^* \phi$ iff $\{q \leq p \mid q \Vdash^* \phi\}$ is dense below p.

More Jargon

Proof: If $\phi = (\sigma = \sigma)$, let $x \in dom(\sigma)$. This means, $\forall p \forall q \leq p[\forall x \in dom(\sigma)(x \in \sigma \leftrightarrow x \in \sigma)]$ as it is a tautology. Hence $p \Vdash^* (\sigma = \sigma)$ for any p.

Now, if $\phi = (x \in \tau)$ where $(x, r) \in \tau$ and $p \Vdash^* x \in \tau$, $r \geq p$. Now, if $x \in dom(\tau)$, $\{q \leq p \mid \exists(y, r') \in \tau[q \leq r' \wedge q \Vdash^* y = x]\}$ is dense below p as $r \geq p$, and $q \Vdash^* x = x$ for any q.

For the case of $\sigma = \tau$, it is obvious. For the case of $\sigma \in \tau$, $\{q \leq p \mid \exists(x, r) \in \tau[q \leq r \wedge q \Vdash^* x = \sigma]\}$ is dense below p. Now, let $s \leq p$, then $\{q \leq s \mid \exists(x, r) \in \tau[q \leq r \wedge q \Vdash^* x = \sigma]\}$ is dense below s.

Proof(contd...):($\rightarrow$) This direction follows directly from the previous result. ($\leftarrow$) Let $\phi = (\sigma \in \tau)$, and the set $\{q \leq p \mid q \Vdash^* \sigma \in \tau\}$ is dense below p. Now, let $s \leq p$, then there exists $s' \leq s$ s.t. $s' \Vdash^* \sigma \in \tau$. But this means there exists $s'' \leq s'$ s.t. $s'' \Vdash^* \sigma = \nu$, where $(\nu, r) \in \tau$ and $r \geq s''$. Therefore, the set $\{q \leq p \mid \exists(\nu, r) \in \tau[r \geq q \wedge q \Vdash^* \sigma = \nu]\}$ is dense below p and hence $p \Vdash^* \sigma \in \tau$. Now let $\phi = (\sigma = \tau)$, the set $\{q \leq p \mid q \Vdash^* \sigma = \tau\}$ is dense below p. Now, let $p \Vdash^* x \in \sigma$ for some $x \in dom(\sigma)$. To see that $p \Vdash^* x \in \tau$, observe $\forall q \leq p[q \Vdash^* x \in \sigma]$ and as $\{q \leq p \mid q \Vdash^* \sigma = \tau\}$ is dense, $\{q \leq p \mid q \Vdash^* x \in \tau\}$ is dense too. And hence, $p \Vdash^* x \in \tau$ from the previous result. As x was arbitrary in $dom(\sigma)$, we get $p \Vdash^* \sigma = \tau$.

Proof(contd...): Now, for complex ϕ in the 3rd and the 4th problem, we induct on the complexity of formulas. If the formulas are of complexity 0, then the previous results give us the base case. Now, let's assume the third and fourth propositions hold for formulas of complexity of $< n$. Now, let ϕ be of complexity n. Then we have the following three cases:

- $\phi = \psi \wedge \theta$. Now, let $p \in \mathbb{P}$, if $p \Vdash^* \phi$ then $p \Vdash^* \psi$ which gives us $q \Vdash^* \psi$ for all $q \leq p$ by I.H. and same for θ. Therefore, $q \Vdash^* \phi$ whenever $q \leq p$ from definition. Now, $(\rightarrow)$ is evident from the previous result and for the $(\leftarrow)$, let $\{q \leq p \mid q \Vdash^* \phi\}$ be dense below p. Therefore, by definition, $\{q \leq p \mid q \Vdash^* \psi\}$ is also dense below p and so is $\{q \leq p \mid q \Vdash^* \theta\}$ and hence by I.H. $q \Vdash^* \psi$ and $q \Vdash^* \theta$. The result follows from definition.

Proof(contd...):

- $\phi = \exists x \psi(x)$. Let $p \Vdash^* \phi$, then the set $\{q \leq p \mid \exists \tau \in V^{\mathbb{P}}[q \Vdash^* \psi(\tau)]\}$ is dense below p. If a set is dense below p and $s \leq p$, then the same set dense below s. Therefore, the set $\{q \leq s \mid \exists \tau \in V^{\mathbb{P}}[q \Vdash^* \psi(\tau)]\}$ is dense below for any $s \leq p$. This proves the claim. $(\rightarrow)$ is trivial here, now to prove $(\leftarrow)$, assume $\{q \leq p \mid q \Vdash^* \phi\}$ is dense below p. Now, let $s \leq p$, so there exists $s' \leq s$ s.t. the set $\{q \leq s' \mid \exists \tau \in V^{\mathbb{P}}[q \Vdash^* \psi(\tau)]\}$ is dense below s'. So, finally we get a $s'' \leq s'$ s.t. $s'' \Vdash^* \psi(\tau)$ for some $\tau \in V^{\mathbb{P}}$. By transitivity, $s'' \leq s$. Therefore, the set $\{q \leq p \mid \exists \tau \in V^{\mathbb{P}}[q \Vdash^* \psi(\tau)]\}$, is dense below p. This proves our result.
- $\phi = \neg\psi$. Let $p \Vdash^* \phi$ and $q \nVdash^* \phi$ for some $q \leq p$. But from definition, that means $\exists q' \leq q[q' \Vdash^* \psi]$ but by transitivity it says $\exists q' \leq p[q' \Vdash^* \psi]$, which implies $p \nVdash^* \phi$ from definition, a contradiction. Again $(\rightarrow)$ follows, to see $(\leftarrow)$, observe that if $p \nVdash^* \phi$, by definition $\exists q \leq p[q \Vdash^* \psi]$. Now, by I.H., $q' \Vdash^* \psi$ for any $q' \leq q$. Hence the set $\{q \leq p \mid q \Vdash^* \phi\}$ is not dense below p as $q \leq p$. By contraposition we get the result.

Much Ado about Nothing...?

But why do we care about this relation at all?

Contents

Models at Last

Finally, returning to $M[G]$, we define another relation $\Vdash$.

definition

Let $p \in \mathbb{P}$, G be a generic filter of $\mathbb{P}$ and ϕ be a formula of $\mathcal{FL}_{\mathbb{P}} \cap M$ i.e the class of formulas with free variables replaced by $\mathbb{P}$-names from $M^{\mathbb{P}}$. We say $p \Vdash \phi$ iff $M[G] \models \phi$ whenever $p \in G$.

In the previous definition τ should be interpreted as τ_G and $\in$ as the inclusion relation.

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In the previous definition τ should be interpreted as τ_G and $\in$ as the inclusion relation. The relation $\Vdash$ is known as the forcing relation.

Models at Last

remark

- Notice in the previous definition, ϕ is a **sentence**, in the extended language $\{\in, M^{\mathbb{P}}\}$
- The relation $M[G] \models \phi$ is a ternary relation.

Let G be any generic filter and let

$$S = \{\phi \in \mathcal{FL}_{\mathbb{P}} \cap M \mid M[G] \models \phi\}$$

and

$$S' = \{\phi \in \mathcal{FL}_{\mathbb{P}} \cap M \mid \exists p \in G(p \Vdash \phi)\}$$

From definition of $\Vdash$ it follows that $S' \subseteq S$,

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Lemma

Let G be a generic filter and ϕ be any formula in $\mathcal{FM}_{\mathbb{P}} \cap M$. If $M[G] \models \phi$ then there exists $p \in G$ s.t. $p \Vdash \phi$.

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Intuitively, the previous Lemma just tells us how the sentences and the elements of $\mathbb{P}$ are tangled together while defining $M[G]$.In fact, as it will be apparent in the section slides, the properties of $M[G]$ gets decided the moment G gets chosen. And the proof gives us the answer to the second question.

Tying things together

Lemma

Let G be a generic filter and $\phi \in \mathcal{FL}_{\mathbb{P}} \cap M$. Then,

- If $p \in G$ and $(p \Vdash^* \phi)^M$, then $M[G] \models \phi$
- If $M[G] \models \phi$, then there exists $p \in G$, s.t. $(p \Vdash^* \phi)^M$.

Proof: Firstly proving the lemma for atomic formulas. We do so by induction on R.

Let $\phi = \sigma \in \tau$. Let $p \in G$, and $p \Vdash^* \phi$. Then the set $D = \{q \leq p \mid \exists(\nu, r) \in \tau(q \leq r \wedge q \Vdash^* \sigma = \nu)\}$ is dense below p. Choose $s \in G \cap D$ and $s \Vdash^* \sigma = \nu$. But $M[G] \models \nu \in \tau$, and by I.H. $M[G] \models \sigma = \nu$. Therefore, $M[G] \models \sigma \in \tau$.

Let $\phi = (\sigma = \tau)$. Now, if $p \Vdash^* (\sigma = \tau)$ and $p \in G$, by definition of $\Vdash^*$ we get that $\forall q \leq p \forall x \in (dom(\sigma) \cup dom(\tau))[q \Vdash^* x \in \sigma \leftrightarrow q \Vdash^* x \in \tau]$. Now, if $M[G] \models x \in \sigma$, that means $x_G \in \sigma_G$, i.e. $(x', r) \in \sigma$ where $r \in G$ and $x'_G = x_G$.

Proof(contd...): Let $q \in G$ s.t. $q \leq p$ and $q \leq r$. Therefore, $q \Vdash^* x \in \sigma$ from definition of $\Vdash^*$ and $q \Vdash^* x' \in \tau$. Therefore by I.H., $M[G] \models x' \in \tau$, but $x'_G = x_G$. So, we proved $M[G] \models \sigma \subseteq \tau$, the opposite direction is exactly the same. Therefore, $M[G] \models (\sigma = \tau)$.

For the 2nd part, we still use induction on R and let $\phi = \sigma \in \tau$. Let $M[G] \models \sigma \in \tau$ then $(\nu, r) \in \tau$ where $r \in G$ and $\nu_G = \sigma_G$, so $M[G] \models \nu = \sigma$. Therefore, by I.H. there exists some $p \Vdash^* \nu = \sigma$, now as G is a filter, we get $q \in G$ s.t. $q \leq r$ and $q \leq p$. Therefore, $\forall q' \leq q \exists (x, q'') \in \tau [q' \leq q'' \wedge q' \Vdash^* x = \sigma]$ by I.H.. By definition of $\in$ in $\Vdash^*$, $q \Vdash^* \sigma \in \tau$.

Let $\phi = (\sigma = \tau)$. Now, let $M[G] \models \sigma = \tau$, if there is some $p \Vdash^* \sigma = \tau$, then we are done. If not, then for all $p \in \mathbb{P}$, $p \nVdash^* \sigma = \tau$. Therefore the set

$$D = \{p \in \mathbb{P} \mid \exists \nu \in (dom(\sigma) \cup dom(\tau))[(p \Vdash^* \nu \in \sigma \wedge p \nVdash^* \nu \in \tau$$

$$\vee (p \Vdash^* \nu \in \tau \wedge p \nVdash^* \nu \in \sigma)]\}$$

is dense. Now, let $q \in G \cap D$. Then, assume $q \Vdash^* \nu \in \sigma$ and $q \nVdash^* \nu \in \tau$, then by previous result $M[G] \models \nu \in \sigma$. Now, as $M[G] \models \nu \in \sigma$, this implies $M[G] \models \nu \in \tau$, by I.H. there exists $q' \in G$ s.t. $q' \Vdash^* \nu \in \tau$. But there exist $r \leq q$ and $r \leq q'$, a contradiction. Same follows for the other condition. This proves there must exist such a p.

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Note

In the proof we have used $(p \Vdash^* \phi)^M$ as $p \Vdash^* \phi$. It was earlier said that $\Vdash^*$ is a relation, but $\Vdash^*$ is not absolute over arbitrary ϕ. Although, for atomic ϕ, it is absolute.

Note

In the proof we have used $(p \Vdash^* \phi)^M$ as $p \Vdash^* \phi$. It was earlier said that $\Vdash^*$ is a relation, but $\Vdash^*$ is not absolute over arbitrary ϕ. Although, for atomic ϕ, it is absolute.

Lemma

Let $p \in \mathbb{P}$, and $\phi \in \mathcal{FL}_{\mathbb{P}} \cap M$. Then $p \Vdash^* \phi$ iff $p \Vdash \phi$.

Proof: $(\rightarrow)$ It is the first condition of the previous lemma.

($\leftarrow$) Suppose $p \Vdash \phi$ and $\phi \not\Vdash^* \phi$. Then, if $\neg\exists q \leq p(q \Vdash^* \neg\phi)$, then $p \Vdash^* \phi$. Therefore there exists $q \leq p$ with $q \Vdash^* \neg\phi$. Consider any G, where G is generic, with $q \in G$. By previous lemma, $M[G] \models \neg\phi$ but $p \Vdash \phi$ gives us a contradiction.

($\leftarrow$) Suppose $p \Vdash \phi$ and $\phi \nVdash^* \phi$. Then, if $\neg\exists q \leq p(q \Vdash^* \neg\phi)$, then $p \Vdash^* \phi$. Therefore there exists $q \leq p$ with $q \Vdash^* \neg\phi$. Consider any G, where G is generic, with $q \in G$. By previous lemma, $M[G] \models \neg\phi$ but $p \Vdash \phi$ gives us a contradiction.

Replacing $\Vdash^*$ by $\Vdash$ gives us the truth Lemma.

Cooking up names

It's natural to ask the question: Can we have the forcing relation $\Vdash$ within M?

Cooking up names

It's natural to ask the question: Can we have the forcing relation $\Vdash$ within M? The answer is NO! This would violate Tarski's undefinability theorem. Therefore, given a $\mathbb{P}$, we can't really have a predicate ψ such that

$$\Vdash = \{(p, \ulcorner\phi\urcorner) \mid p \in \mathbb{P} \wedge \tau_1, ..., \tau_n \in V^{\mathbb{P}} \wedge \psi(p, \ulcorner\phi\urcorner, \tau_1, ..., \tau_n)\}$$

But fortunately, we have the next best thing.

Who let him cook?

Let $\phi(x_1, ..., x_n)$ be a formula with the variables shown and $\mathbb{P}$ be a forcing poset.

definition

Let $\phi(x_1, ..., x_n)$ be a $\{\in\}-$formula with the variables shown and $\mathbb{P}$ be a forcing poset. Then

$$\{(p, \tau_1, ..., \tau_n) \mid p \in \mathbb{P} \wedge \tau_1, ..., \tau_n \in M \wedge p \Vdash \phi(\nu_1, ..., \nu_n)\}$$

is definable in M without any parameters.

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This is known as the definability lemma.
Proof: If the lemma holds for atomic ϕ, it can be extended to all the ϕs in the set of formulas quite easily.

Sorry :(

But wait, how do you prove it for atomic formulas?

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But wait, how do you prove it for atomic formulas?Well note that atomic formulas are defined "recursively", maybe there is something going on there. As M is a set model, we use Gödel encoding to do something.

Frame Title

So, how much "control" do we actually have over $M[G]$? Can we actually force weird stuff?

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Let $\mathbb{P} = Fn(\omega, 2^\omega)$, therefore $\bigcup G$ is function from ω onto 2^ω. AND, $\bigcup G \in M[G]$. So, is $M[G]$ not a model, or is ZFC inconsistent?

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The answer is neither!

Magic!

Notice that ω and 2^ω are both in M and 2^ω is NOT ABSOLUTE. So, even though 2^ω appears as a set in $M[G]$ it doesn't represent the "meaning" of 2^ω. In general, while defining $\cup G$, we ended up adding subsets of ω.
Consider the set

$$S := \{n \in \omega \mid n \notin \cup G(n)\}$$

So, even though forcing is a strong technique it can do "weird" things.

Thank You!