Homework 10, due Tuesday 27 May

1. Prove Theorem 22 (p 21 of the syllabus) for the system of natural deduction.

   The clause "$\Gamma \,|\, \bot$ iff $\Gamma \vdash_{\mathbf{IPC}} \bot$" should be added to Definition 21. We noted in class that a corollary of the definition is that $\Gamma \,|\, \varphi$ implies $\Gamma \vdash_{\mathbf{IPC}} \varphi$. You can use this.

   Also, you can use the following:

   Lemma: If Γ and Σ are two **IPC**-equivalent theories (i.e. $\Gamma \vdash_{\mathbf{IPC}} \sigma$ for all $\sigma \in \Sigma$ and $\Sigma \vdash_{\mathbf{IPC}} \gamma$ for all $\gamma \in \Gamma$), then for all χ: $\Gamma|\chi$ iff $\Sigma|\chi$.

   Hint: you should not use induction on the complexity of a single derivation $\Gamma \vdash \phi$, but rather induction on n, proving the following statement: "for all Σ and all ψ, if $\Sigma|\Sigma$ and $\Sigma \vdash_n \psi$ then $\Sigma|\psi$", where "$\Sigma \vdash_n \psi$" means that the derivation-tree for $\Sigma \vdash \psi$ has depth n or less. So you need to assume that the statement "for all Σ and all $\psi \ldots$" holds for all $m < n$ and prove that "for all Σ and all $\psi \ldots$" holds for n. [6 pts]

2. (a) Show Corollory 23 (1) (completely, even if in class we did some of the steps). [2 pts]

   (b) Show how Corollory 23 (2) follows from Corollory 23 (1). [2 pts]

   (c) Show, using the Aczel slash, Exercise 2 (a) from Homework 5, namely that

   $$\text{If } \vdash_{\mathbf{IPC}} (\varphi \to \psi) \to (\chi \vee \theta), \text{ then}$$
   $$\vdash_{\mathbf{IPC}} (\varphi \to \psi) \to \chi \text{ or } \vdash_{\mathbf{IPC}} (\varphi \to \psi) \to \theta \text{ or } \vdash_{\mathbf{IPC}} (\varphi \to \psi) \to \varphi.$$

   [2 pts]

3.* Show that, if θ has n propositional variables, then $\theta \,|\, \theta$ iff in the n-universal model the following holds:

   $$\text{For all } u, v, \text{ if } u \models \theta \text{ and } v \models \theta$$
   $$\text{then there exists a } w \text{ with } wRu \text{ and } wRv \text{ such that } w \models \theta.$$

   [6 pts]