Homework 9, due Monday 2 May

1. Draw all 3-element intuitionistic frames, and all 4-element rooted intuitionistic frames. Draw all 5-element Heyting-algebras, and indicate their join-prime elements. Everything up to isomorphism; no proofs needed.

   [5 pts]

2. (a) Show that, if $\mathfrak{F}$ is rooted, then $\Psi(\mathfrak{F})$ has a second-largest element. [2 pts]

   (b) Show that, if $\mathfrak{A}$ has a second-largest element, then $\Phi(\mathfrak{A})$ is rooted. [2 pts]

3. A relation $\equiv$ is called a *congruence* on a Heyting-algebra $\mathfrak{A}$ if it is an equivalence relation and for all $a, a', b, b' \in \mathfrak{A}$, if $a \equiv a'$ and $b \equiv b'$, then $(a \star b) \equiv (a' \star b')$ for all three operations $\star$ on $\mathfrak{A}$.

   (a) Show that, if F is a filter on $\mathfrak{A}$, then $\equiv_F$, defined by $a \equiv_F a'$ iff $(a \leftrightarrow a') \in F$, is a congruence on $\mathfrak{A}$. (You can use that $\vdash_{\mathbf{IPC}} \varphi$ iff "$\varphi = \top$" is valid in all Heyting-algebras, and you can just assume that $\vdash_{\mathbf{IPC}} \varphi$ holds for φ if that is the case.) [3 pts]

   Denote by $\|a\|_F$ the equivalence class of a under $\equiv_F$. On the set $\|A\|_F$ of all these equivalence classes define $\|a\|_F \star \|a'\|_F := \|a \star a'\|_F$ for all operations $\star$, and $\bot := \|\bot\|_F$. The resulting algebra $\mathfrak{A}/F$ is called the *quotient algebra of* $\mathfrak{A}$ *with respect to* F.

   (b) Suppose that h is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. Show that the map g, defined by $g(h(a)) := \|a\|_{h^{-1}(\top)}$, is an isomorphism of $\mathfrak{B}$ onto $\mathfrak{A}/h^{-1}(\top)$. [3 pts]

   (c) Suppose that F is a filter on a Heyting-algebra $\mathfrak{A}$. Show that the map h, defined by $h(a) := \|a\|_F$, is an homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/F$, and $F = h^{-1}(\top)$. [3 pts]