Homework 4, due Monday 28 February.

1. (a) Let φ contain only $\wedge, \vee$ and $\to$ but no $\neg$ and no $\bot$. Let $\mathfrak{M}$ be any Kripke-model (for the language of φ). Extend the model $\mathfrak{M}$ to $\mathfrak{M}^+$ by adding one more node x at the top above all the nodes of $\mathfrak{M}$, and making all the propositional variables of φ true in x.

   Show that, for all the nodes w in $\mathfrak{M}$ we have:

   $$\mathfrak{M}, w \models \varphi \text{ iff } \mathfrak{M}^+, w \models \varphi$$

   (satisfaction in the old and new model is the same for φ). [4 pts]

   (b) Let φ contain only $\wedge, \vee$ and $\to$ but no $\neg$ and no $\bot$. Show that $\vdash_{\mathbf{IPC}} \varphi$ iff $\vdash_{\mathbf{KC}} \varphi$. (You may use the completeness of **KC** with respect to its finite frames.) [2 pts]

2. (a) Prove, using the canonical model method, strong completeness of **LC** with respect to the upwards linear frames. [2 pts]

   (b) Prove, using the canonical model method, strong completeness of **KC** with respect to the upwards directed frames. [3 pts]

3. The *n-canonical model* is the canonical model for formulae in the n variables $p_1, \ldots, p_n$ only. Prove that the n-canonical model of $\mathbf{KC_{prop}} := \mathbf{IPC} + \{\neg p \vee \neg\neg p\}$ for propositional variables p only, has a largest element above each world, and simultaneously that $\mathbf{KC_{prop}}$ is equal to **KC**, in the following manner:

   (a) Take a node Γ of the n-canonical model of $\mathbf{KC_{prop}}$. Show that Γ contains either $\neg p_i$ or $\neg\neg p_i$ for each $i \leq n$. [1 pt]

   (b) Assume without loss of generality that Γ contains $\neg p_1, \ldots, \neg p_k$ and $\neg\neg p_{k+1}, \ldots, \neg\neg p_n$. Show that then $\Gamma \cup \{\neg p_1, \ldots, \neg p_k, p_{k+1}, \ldots, p_n\}$ is consistent. [2 pts]

   (c) Show that it follows that Γ has a successor containing $\{\neg p_1, \ldots, \neg p_k, p_{k+1}, \ldots, p_n\}$ and this successor is the unique largest element of the model above Γ. (You may use the fact that $\mathbf{CPC_{prop}} := \mathbf{IPC} + \{p \vee \neg p\}$ for propositional variables p only, is equal to **CPC**). [2 pts]

   (d) Conclude that any such Γ satisfies all of **KC** and therefore $\mathbf{KC_{prop}}$ is equal to **KC**. [2 pts]

   Remark: If you feel like it, you can show that with a little additional work you can use the full canonical model instead of the n-canonical model, for some bonus points.