Forcing ¬CH

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Introduction

We want to construct a model for ZFC + ¬CH.

Proper class models cannot do the trick (as we will show).

Idea: Extend a *set model* so that CH is false.

Recap:

1. Take a countable transitive model M for ZFC.
2. Take a forcing poset $(\mathbb{P}, \leq, \mathbb{1}) \in M$ and a $\mathbb{P}$-generic filter G.
3. Extend M to a larger model $M[G]$ for ZFC that contains G.

We need to choose an appropriate $\mathbb{P}$ that forces $M[G] \models \neg$CH.

Once we have such a $\mathbb{P}$, we obtain Con(ZFC)$\rightarrow$ Con(ZFC + ¬CH).

Introduction

Caution: But how did we get a set model M?

Two solutions:

- *Inaccessible cardinals*. Use that $V_\kappa \models$ ZFC.
- *Finite fragments.* If ¬Con(ZFC $+$ ¬CH) there exists some finite $\Omega \subseteq$ ZFC such that $\Omega + \neg$CH $\vdash \bot$. Then in ZFC, we can prove the existence of a ctm $M[G]$ for $\Omega + \neg$CH, starting from a ctm M for ZFC. Yet again, this proof only uses that M satisfies some finite fragment ZFC*, and the Reflection Theorem provides a ctm for ZFC*.

Overview

Proper class models fail

Lemma. Suppose that in ZF, we can construct a transitive proper class model for ZFC $+ \neg$CH. Then ZF is inconsistent.

Proof. Suppose we have constructed such a transitive proper class M in ZF. Then in particular, M can be constructed in ZFC $+ (V = L)$. The axiom $V = L$ then implies $M \subseteq L$.

However, since M is a proper class and the rank function is absolute for transitive models, we must have $ON \subseteq M$. Recalling that the L_α-hierarchy is absolute for transitive models, we obtain $L \subseteq M$.

Thus $M = L$, but then $M \models$ CH. However, we assumed $M \models \neg$CH, so ZFC $+ (V = L)$ is inconsistent, which in turn shows that ZF is inconsistent. □

The forcing poset $\mathrm{Fn}(I, J)$

For sets I, J we define $\mathrm{Fn}(I, J)$ as the set of all finite partial functions from I to J. For $f, g \in \mathrm{Fn}(I, J)$ we write $f \leq g$ iff $f \supseteq g$. We always have $\emptyset \in \mathrm{Fn}(I, J)$ and we take $\mathbb{1} = \emptyset$.

$(\mathrm{Fn}(I, J), \leq, \mathbb{1})$ is a forcing poset.

Note that f extends g in the forcing poset precisely when f extends g as a function.

If M is a ctm for ZFC and $I, J \in M$ then $(\mathrm{Fn}(I, J), \leq, \mathbb{1}) \in M$ by absoluteness.

The Delta System Lemma

Lemma. Let κ be an uncountable regular cardinal, and let $\mathcal{A}$ be a family of finite sets with $|\mathcal{A}| = \kappa$. Then there exists a *delta system* $\mathcal{B} \subseteq \mathcal{A}$ of size κ with a finite *root* R, that is we have

$$X \cap Y = R \text{ for all distinct } X, Y \in \mathcal{B}.$$

Proof. κ is regular and $\mathcal{A} = \bigcup_{n\in\omega}\{X \in \mathcal{A} \colon |X| = n\}$ has size κ. Therefore there must be an $n \in \omega$ such that $\{X \in \mathcal{A} \colon |X| = n\}$ has size κ. Without loss of generality we may assume that each $X \in \mathcal{A}$ has size n.

We use induction on $n > 0$. Note $n = 0$ does not occur.

For $n = 1$, the statement is trivial.

The Delta System Lemma

Suppose $n > 1$. Define $\mathcal{A}_t = \{X \in \mathcal{A} \colon t \in X\}$ for all t.

Two cases:

1. Suppose $|\mathcal{A}_t| < \kappa$ for all t. Then for any S with $|S| < \kappa$, the set $\{X \in \mathcal{A} \colon X \cap S \neq \emptyset\} = \bigcup_{t \in S} \mathcal{A}_t$ is smaller than κ, therefore $X \cap S = \emptyset$ for some $X \in \mathcal{A}$.

   Thus we can recursively define $\langle X_\alpha \in \mathcal{A} \colon \alpha \in \kappa \rangle$ such that for every $\alpha \in \kappa$ we have $X_\alpha \cap \bigcup_{\beta < \alpha} X_\beta = \emptyset$.

   Take $\mathcal{B} = \{X_\alpha \colon \alpha \in \kappa\}$ and $R = \emptyset$.

2. Suppose $|\mathcal{A}_t| = \kappa$ for some t. Using the induction hypothesis on $\mathcal{C} = \{X \setminus \{t\} \colon X \in \mathcal{A}_t\}$ we obtain a delta system $\mathcal{D} \subseteq \mathcal{C}$ with root T.

   Take $\mathcal{B} = \{Z \cup \{t\} \colon Z \in \mathcal{C}\}$ and $R = T \cup \{t\}$. □

The forcing poset $\mathrm{Fn}(I, J)$

Lemma. $\mathrm{Fn}(I, J)$ has the ccc iff $I = \emptyset$ or J is countable.

Proof. If I or J is empty then $\mathrm{Fn}(I, J) = \{\emptyset\}$ which is ccc. Otherwise:

- $\Rightarrow$ If J is uncountable then fix an $x \in I$. Now the singleton functions $\{(x, y)\}$ for $y \in J$ form an uncountable antichain.
- $\Leftarrow$ If J is countable suppose we have $\langle p_\alpha \colon \alpha \in \omega_1 \rangle$ in $\mathbb{P}$.

  By the Delta System Lemma there exists an uncountable $B \subseteq \omega_1$ and a finite root $R \subseteq I$ such that for any $\alpha, \beta \in B$ with $\alpha \neq \beta$ we have $\mathrm{dom}(p_\alpha) \cap \mathrm{dom}(p_\beta) = R$.

  Since J^R is countable, there exist $\alpha, \beta \in B$ with $\alpha \neq \beta$ and $p_\alpha \restriction R = p_\beta \restriction R$. But then $p_\alpha \not\perp p_\beta$ so the sequence is not an antichain. □

Preservation of Cardinals

Let M be a ctm for ZFC.

Definition. A forcing poset $\mathbb{P}$ preserves cardinals iff for all generic G and $\alpha \in \mathrm{o}(M)$ we have: $(\alpha \text{ is a cardinal})^M$ iff $(\alpha \text{ is a cardinal})^{M[G]}$.

Theorem. If $(\mathbb{P} \text{ is ccc})^M$ then $\mathbb{P}$ preserves cardinals.

Lemma. A forcing poset $\mathbb{P}$ preserves cardinals iff for all generic G and $\alpha \in \mathrm{o}(M)$ we have $(\aleph_\alpha)^M = (\aleph_\alpha)^{M[G]}$.

Note: $(\aleph_\alpha)^M = (\aleph_\alpha)^{M[G]}$ does not imply $(2^{\aleph_\alpha})^M = (2^{\aleph_\alpha})^{M[G]}$.

Preservation of Cardinals

Lemma. A forcing poset $\mathbb{P}$ preserves cardinals iff for all generic G and $\alpha \in \mathsf{o}(M)$ we have $(\aleph_\alpha)^M = (\aleph_\alpha)^{M[G]}$.

Proof.

- $\Leftarrow$ Because every infinite cardinal can be written as $\aleph_\alpha$.
- $\Rightarrow$ By induction on $\alpha \in \mathsf{o}(M)$.

  Assume $(\aleph_\alpha)^M = (\aleph_\alpha)^{M[G]}$. We see $(\aleph_{\alpha+1})^M \leq (\aleph_{\alpha+1})^{M[G]}$ because $M \subseteq M[G]$. However $(\aleph_{\alpha+1})^M$ is also a cardinal in $M[G]$. Therefore $(\aleph_{\alpha+1})^M = (\aleph_{\alpha+1})^{M[G]}$.

  Assume α is a limit and for all $\beta < \alpha$ that $(\aleph_\beta)^M = (\aleph_\beta)^{M[G]}$. Now $(\aleph_\alpha)^M = \bigcup_{\beta<\alpha}(\aleph_\alpha)^M = \bigcup_{\beta<\alpha}(\aleph_\beta)^{M[G]} = (\aleph_\beta)^{M[G]}$. □

Forcing $\neg\mathrm{CH}$

We are now ready to give a model for ZFC + $\neg$CH.

1. Let M be a ctm for ZFC and let $\gamma \in \mathrm{o}(M)$. Write $\kappa = (\aleph_\gamma)^M$.
2. Let $\mathbb{P}$ denote the forcing poset $\mathrm{Fn}(\kappa \times \omega, 2)$ and let G be a $\mathbb{P}$-generic filter over M. Since $\kappa \times \omega, 2 \in M$ we have $(\mathbb{P}, \supseteq, \emptyset) \in M$.
3. We obtain a ctm $M[G]$ for ZFC with $M \subseteq M[G]$, $G \in M[G]$.

We show $M[G] \models \neg$CH by constructing an injection from $\aleph_\gamma$ to 2^ω *within* $M[G]$, which gives $M[G] \models 2^{\aleph_0} \geq \aleph_\gamma$.

That is, we construct an injection from $(\aleph_\gamma)^{M[G]}$ to $(2^\omega)^{M[G]}$ that lives in $M[G]$.

Forcing $\neg\mathrm{CH}$

As G is a filter we have that $f_G := \bigcup G$ defines a partial function.

For each $i \in \kappa \times \omega$, absoluteness gives

$$D_i := \{q \in \mathbb{P} \colon i \in \mathrm{dom}(q)\} \in M.$$

Each D_i is *dense*: any partial function can be extended to one with i in its domain. So G intersects every D_i and thus $f_G \colon \kappa \times \omega \to 2$.

Then f_G defines a sequence $\langle h_\alpha \colon \alpha \in \kappa \rangle$ of functions

$$\begin{aligned} h_\alpha \colon \omega &\to 2, \\ n &\mapsto f_G(\alpha, n). \end{aligned}$$

Note f_G is in the extended model $M[G]$ since $G \in M[G]$, so the sequence $\langle h_\alpha \colon \alpha \in \kappa \rangle$ is in $M[G]$ as well.

Forcing $\neg\mathrm{CH}$

We will show that the h_α are distinct.

For $\alpha, \beta \in \kappa$ with $\alpha \neq \beta$, define $E_{\alpha,\beta}$ as the set

$$\{q \in \mathbb{P} \colon \exists n \in \omega[(\alpha, n), (\beta, n) \in \mathrm{dom}(q) \wedge q(\alpha, n) \neq q(\beta, n)]\}.$$

By absoluteness, each $E_{\alpha,\beta}$ is in M.

Note that each $E_{\alpha,\beta}$ is dense: for any $p \in \mathbb{P}$ there exists an $n \in \omega$ with $(\alpha, n), (\beta, n) \notin \mathrm{dom}(p)$, so we can extend p to a $q \in E_{\alpha,\beta}$ with

$$q \colon \mathrm{dom}(p) \cup \{(\alpha, n), (\beta, n)\} \to 2.$$

So there exists a $q \in E_{\alpha,\beta} \cap G$ which implies there is an $n \in \omega$ with

$$h_\alpha(n) = f_G(\alpha, n) = q(\alpha, n) \neq q(\beta, n) = f_G(\beta, n) = h_\beta(n).$$

Forcing $\neg\mathrm{CH}$

Thus we obtain an injection $h \in M[G]$ given by

$$h\colon \kappa \to (2^\omega)^{M[G]},$$
$$\alpha \mapsto h_\alpha.$$

Recall $\kappa = (\aleph_\gamma)^M$.

Because $(2 \text{ is countable})^M$ we have $(\mathbb{P} \text{ is a ccc})^M$. Therefore $\mathbb{P}$ preserves cardinals, and thus $\kappa = (\aleph_\gamma)^M = (\aleph_\gamma)^{M[G]}$.

So we have our injection from $(\aleph_\gamma)^{M[G]}$ to $(2^\omega)^{M[G]}$, showing

$$M[G] \models 2^{\aleph_0} \geq \aleph_\gamma.$$

In particular we can take $\gamma = 2$ in which case

$$M[G] \models \mathsf{ZFC} + \neg\mathsf{CH}.$$