Details on Reflection Theorems

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January 29, 2021

Overview

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1. **Lemma**: a Condition for Absoluteness
2. **Theorem 1**: a countable non-transitive model
3. **Theorem 2**: an (uncountable) transitive model
4. **Theorem 3**: a countable transitive model
5. Relevance to Forcing

The Lemma - a Condition for Absoluteness

Lemma (II.5.2 in Kunen 2011 and II.7.2 in Kunen 1980)

For any finite subformula-closed *list of formulae* $\varphi_1, \ldots, \varphi_n$*, and any two classes* $\emptyset \subseteq M \subseteq N$*, the following are equivalent:*

1. $\bigwedge_{i \leq n} M \preccurlyeq_{\varphi_i} N$
   Equivalently: $\varphi_1, \ldots, \varphi_n$ *are absolute for* M, N
2. *For each existential formula* φ_i *of the form* $\exists x \varphi_j(x, \overrightarrow{y})$*, the following holds:*

$$\forall \overrightarrow{m} \in M \Big(\exists x \in N : \varphi_i^N(x, \overrightarrow{m}) \rightarrow \exists x \in M : \varphi_j^N(x, \overrightarrow{m}) \Big)$$

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Note: *subformula-closed* lists, *absoluteness* for two classes, and *relativisation* of a formula.

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(!) Any finite list closed under subformulae will still be finite.

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Definition (Absoluteness for M, N)

Let $M \subseteq N$ be classes. A formula φ with $x_1, \ldots, x_n$ free variables is absolute for M, N iff

$$\forall x_1, \ldots, x_n \in M \Big(\varphi^M(x_1, \ldots, x_n) \leftrightarrow \varphi^N(x_1, \ldots, x_n) \Big)$$

Lemma

1. $\varphi_1, \ldots, \varphi_n$ *are absolute for* M, N
2. *For all* $\exists x \varphi_j(x, \overrightarrow{y})$*:* $\forall \overrightarrow{m} \in M \Big(\exists x \in N : \varphi_j^N(x, \overrightarrow{m}) \rightarrow \exists x \in M : \varphi_j^N(x, \overrightarrow{m}) \Big)$

Proof

- Let M and N be classes such that $M \subseteq N$
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- Use absoluteness of φ_i and its subformula φ_j:
  - absoluteness of φ_i gives $\exists x \in M \varphi_j^M(x, y_1, \ldots, y_n)$
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So we find $\varphi_i^M(y_1, \ldots, y_n) \leftrightarrow \varphi_i^N(y_1, \ldots, y_n)$ (absolute for M, N). □

Theorem 1

Model 1: not (always) transitive, but countable

Theorem (1)

Let ZFC^* be a finite fragment of ZFC. For any X there is M such that $X \subseteq M$, $M \models ZFC^*$, and $|M| \leq max(\aleph_0, |X|)$.

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Let ZFC^* be a finite fragment of ZFC. For any X there is M such that $X \subseteq M$, $M \models ZFC^*$, and $|M| \leq max(\aleph_0, |X|)$.

Proof idea. Ensure that condition (2) of the Lemma is satisfied, by adding witnesses in countably many stages.

Proof - Preliminaries

Let $\Phi := \{\varphi_1, \ldots, \varphi_n\}$ be the result of taking ZFC^*, replacing all formulae of form $\forall v \varphi$ with $\neg \exists v \neg \varphi$, and closing under subformulae. Let $X_0 := X$. Let $I := \{i \mid \varphi_i \in \Phi \text{ is an existential formula}\}$, and assume that each existential $\varphi_i(\overrightarrow{x})$ is of the form $\exists y \varphi_j(\overrightarrow{x}, y)$.

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Proof - Ensuring Witnesses

For each $i \in I$, define a function f_i satisfying:

- For every tuple $\overrightarrow{p}$ of sets, if φ_i holds of $\overrightarrow{p}$, then φ_j holds of $(\overrightarrow{p}, f_i(\overrightarrow{p}))$.

(Intuitively: whenever an existential formula holds of a tuple, f_i is a function which finds a witness for that formula).
We need the Axiom of Choice to guarantee that these functions exist!

Proof

Now, for all $k \in \omega$, define recursively:

- $F_{i,k} := \mathrm{Ran}(f_i \restriction \wp(X_k))$
- $F_k := \bigcup_{i \in I} F_{i,k}$ (all the witnesses we need for X_k)
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Then set $M := \bigcup_{k \in \omega} X_k$.

We'll now show that M satisfies each of the desired conditions.
We have $X(= X_0) \subseteq M$ by definition of M.

*M is a model of ZFC**

We show that each $\varphi \in \Phi$ is absolute for M, implying that $M \models ZFC^*$.

Fix an arbitrary existential formula $\varphi_i(x_1, \ldots, x_n)$. Let $\overrightarrow{p} := \{p_1, \ldots, p_n\}$ be arbitrary members of M such that $\varphi_i[p_1, \ldots, p_n]$ holds (in V).

Every member of $\overrightarrow{p}$ appears in some X_k, hence there is some X_m (m the maximum of these ks) containing every member of $\overrightarrow{p}$.

Since $\varphi_i[p_1, \ldots, p_n]$ holds, there is $q = f_i(p_1, \ldots, p_n) \in X_{m+1}$ such that $\varphi_j[p_1, \ldots, p_n, q]$ holds. $X_{m+1} \subseteq M$, hence $q \in M$.

*M is a model of ZFC**

Since φ_i and $\overrightarrow{p}$ were arbitrary, we have shown that for each existential formula φ_i, the following holds:
$\forall \overrightarrow{a} \in M[\varphi_i^V(\overrightarrow{a}) \rightarrow \exists b \in M \psi_i^V(\overrightarrow{a}, b)]$.
By the Lemma, it follows that for every formula of our list and in particular every sentence φ_z of ZFC^*, $M \preccurlyeq_{\varphi_z} V$. But every axiom of ZFC holds in V, and therefore holds in M as well.

$|M| \leq max(|X|, \aleph_0)$.

To see that $|M| \leq max(|X|, \aleph_0)$, distinguish two cases: X is finite, or X is infinite.
Case 1: X finite. If X_k is finite for any fixed k then it is clear that $F_{i,k}, F_k$ will also be finite. Hence, $X_{k+1} = X_k \cup F_k$ will be finite. It follows by induction that X_m is finite for every natural number m. M is therefore a countable union of finite sets, and must therefore be (at most) countable; so $|M| \leq max(|X|, \aleph_0)$.

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Case 2: X infinite. Let κ denote $|X|$, with κ some infinite cardinal. We will show by induction that $|X_m| = \kappa$ for every natural number m. The base case of $m = 0$ is trivial.
Induction step: assume $|X_k| = \kappa$ for some fixed k. For any natural number r, the set of r-tuples $\subseteq X_k$ will have the same cardinality as X_k by Hessenberg's theorem. Then $|F_{i,k}|$ is at most $|X_k| = \kappa$, and since F_k is a finite union of such sets, $|F_k| \leq \kappa$.
Now $X_{k+1} = X_k \cup F_k$ and must therefore have cardinality κ. This completes the induction step, so $|X_n| = \kappa$ for all n.

$|M| \leq max(|X|, \aleph_0)$.

$M = \bigcup_{k\in\omega} X_k$ is a countable union of sets of cardinality κ, and since $\kappa \geq \aleph_0$, it follows that $|M| = \kappa$ as well. Hence again $|M| \leq max(|X|, \aleph_0)$.

So we're done! □

An (uncountable) transitive model

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- Disregard size
- Ensure transitivity at each step

Proof - Preliminaries

Assumptions

Let $\Phi := \{\varphi_1, \ldots, \varphi_n\}$ be subformula-closed

$X_0 := V_\delta$ for least δ such that $V_\delta \supseteq X$

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Note: V_δ is transitive by definition

Proof - Ensuring Witnesses

We again ensure that we have the necessary witnesses.

Strategy:

- find rank α of the witness, rather than witness itself
- Include entire V_α to ensure transitivity

Proof - Recursive Construction

Definition ('Rank' Witness Function)

For some existential formula $\varphi_i = \exists y \varphi_j(\overrightarrow{x}, y)$, let

$$f_i^r(\overrightarrow{p}) = \begin{cases} \text{least } \alpha & \text{such that } \exists y \text{ with } \alpha = rk(y) \text{ and } \varphi_j(\overrightarrow{p}, y) \\ 0 & \text{if there is no such } y \end{cases}$$

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Let $F_{i,k} := \text{Ran}(f_i^r \restriction \wp(X_k))$

Let $\alpha_{k+1} := \bigcup_{i \in I} F_{i,k}$ $\quad (\alpha_0 := \delta)$

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The model: Let $M := \bigcup_{k \in \omega} X_k$.

Proof - Validation

Theorem (2)

For any X there is a transitive M such that $X \subseteq M$ and $M \models ZFC^*$.

- $X \subseteq M$ is trivial.
- $M \models ZFC^*$. As in Theorem 1:
  - Every tuple of free variables in M appears at some X_k
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  - So we satisfy (2) of the Lemma.
  - Using the Lemma, we find $M \preccurlyeq_\varphi V$ for every sentence φ
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□

About the size of M

$M = V_\gamma$ for some $\gamma := sup(\{\alpha_k \mid k \in \omega\})$

At each recursive step, $\alpha_{k+1} > \alpha_k$. So M can grow arbitrarily big. However M will be a set in V as it is bounded under V_γ.

Theorem (3)

If X is transitive, then there is a transitive M such that $X \subseteq M$, $M \models ZFC^*$, and $|M| = max(\aleph_0, |X|)$.

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Proof idea: use Mostowski collapsing function.

Mostowski Collapse - Recap

Let C be a class. Then there is a function π (subscripts omitted) mapping C onto a transitive class T such that $(C, \in) \cong_\pi (T, \in)$. π is called the Mostowski Collapsing function, and is defined by $\pi(x) = \{\pi(z) \mid z \in x \cap C\}$ for all $x \in C$.

Proof

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If X is transitive, then there is a transitive M such that $X \subseteq M$, $M \models ZFC^*$, and $|M| = max(\aleph_0, |X|)$.

Start with X, then use method from theorem 1 to obtain an M with $X \subseteq M$, $M \models ZFC^*$, and $|M| = max(\aleph_0, |X|)$.
Let $\pi(M)$ be the image of M under its Mostowski collapsing function. $\pi(M)$ is transitive by the definition of π.
Since $\pi(M)$ is isomorphic to M, it will satisfy precisely the same formulas as M and is therefore a model of ZFC^*.

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Proof

$X \subseteq \pi(M)$

By assumption, X is transitive.
Suppose for contradiction that there is $x \in X$ such that $\pi(x) \neq x$.
Then there must be an $\in$-least such x.
For this least x, there must be $y \in x$ with $y \notin \pi(x)$. But
$y \in x \in X \implies y \in X \implies y \in M$, and therefore $\pi(y) \in \pi(x)$.
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Proof

$X \subseteq \pi(M)$

We have $\pi(x) = x$ for each $x \in X$.
If $x \in X$ then $x \in M$ and therefore $x = \pi(x) \in \pi(M)$, so every member of X is also a member of $\pi(M)$; i.e. $X \subseteq \pi(M)$. This completes the proof. □

In particular, we can start with $X = \emptyset$. Then X is transitive, so we can obtain a transitive model of ZFC^* with cardinality $\aleph_0$.

Reflection and Independence of $\neg$CH

We show how the Reflection Theorem can be combined with the forcing argument to give a proof that $\neg$CH is relatively consistent with *ZFC*.

Forcing allows us to turn a countable transitive model of ZFC into a model of $ZFC + \neg CH$. But does this assure us that $\neg CH$ is relatively consistent?
The Reflection Theorem did not give us a countable transitive model for all of ZFC, only for a finite subset of the axioms.

How do we solve this problem?

A trick

Extend the language of set theory $\mathcal{L}$ to a new language, $\mathcal{L}^+$, containing two new constant symbols, C and F.

Define Σ to be a set of sentences in $\mathcal{L}^+$, containing each *ZFC* axiom, together with the statements that C is a transitive set and F is a bijection from C into ω, and the relativisation of each *ZFC* axiom to C.

We claim that Σ is a conservative extension of ZFC, meaning if φ is a sentence of $\mathcal{L}$ and $\Sigma \vdash \varphi$, then $ZFC \vdash \varphi$.

To see this, assume $\Sigma \vdash \varphi$ with $\varphi \in \mathcal{L}$. Since proofs are finite, there must be a formula $\rho(x, y)$ such that $ZFC \cup \{\rho(C, F)\} \vdash \varphi$. ($\rho(C, F)$ will say that C is transitive, F bijects ω to C, and asserts some finite conjunction of ZFC axioms relativised to C).

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Lemma

For every $\varphi \in \mathcal{L}$, if $\Sigma \vdash \varphi$ then $ZFC \vdash \varphi$.

Using the Deduction theorem: $ZFC \vdash \rho(C, F) \to \varphi$. Then by the arbitrariness of C, F, can infer $ZFC \vdash \exists x, y \; \rho(x, y) \to \varphi$.
But Reflection 3 tells us that ZFC actually does prove $\exists x, y \; \rho(x, y)$!
Thus, we also have $ZFC \vdash \varphi$.

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Completing the argument

This is very good news, because Σ has all we need to carry out the forcing argument - it asserts that we have a countable transitive model for *ZFC*.

Suppose we want to prove that some specific value - $\aleph_7$, say - is consistent for $2^{\aleph_0}$.
The first step is to turn our C into a ctm that definitely has $\aleph_1$ many reals.

Completing the argument

Define $M := L(o(C))$ and $\mathbb{P} := Fn((\omega_7)^M \times \omega, 2)$. Then define a G which is $\mathbb{P}$-generic over M.

One can then prove $(\varphi)^{M[G]}$ for each φ of ZFC, and also $(2^{\aleph_0} = \aleph_7)^{M[G]}$.

Completing the argument

Finally, if $ZFC + (2^{\aleph_0} = \aleph_7) \vdash 0 = 1$, then we will have $(0 = 1)^{M[g]}$, and hence by absoluteness, $0 = 1$; that is, Σ would prove $0 = 1$.
But since Σ is a conservative extension of ZFC, it would follow that ZFC proves $0 = 1$.
So we have shown that if $ZFC + (2^{\aleph_0} = \aleph_7) \vdash 0 = 1$, then $ZFC \vdash 0 = 1$.

Conclusion

1. **Lemma**: a Condition for Absoluteness
2. **Theorem 1**: a countable non-transitive model
3. **Theorem 2**: an (uncountable) transitive model
4. **Theorem 3**: a countable transitive model
5. Relevance to Forcing

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Thank you!