Forcing and Independence Proofs: Assignment 1

Part A: object language and meta-language

1. For each of the following statements, determine whether they are made in the formal language of set theory $\mathcal{L}_\in$, or in a meta-language (in which we talk *about* set theory). Note that any statement in any meta-language can *in principle* be formalized as a statement in the object language as well. The idea of this exercise is to consider the most natural/obvious meaning.

   (a) Every convergent sequence in $\mathbb{R}$ is bounded from above.

   (b) $\mathsf{ZFC} \vdash$ "Every convergent sequence in $\mathbb{R}$ is bounded from above".

   (c) ZFC contains infinitely many axioms.

   (d) If ZFC is consistent, then $\mathsf{ZFC} + \mathsf{CH}$ is also consistent

   (e) The addition operation on the ordinals is not commutative.

   (f) Every normal function on the ordinals has a fixed point.

   (g) Ord (the class of all ordinals) is not a set.

   (h) There are classes which are not sets.

2. Consider the following informally stated assertion:

   > "For every proper class A and every set X, there exists an injective function $f : X \to A$."

   (a) Write down the above statement formally. You may use the abbreviations "f is a function", "dom(f)" and "ran(f)" without writing them out in detail.

   (b) Is this a statement in the formal language or the meta-language?

   (c) Prove the above assertion (using an informal argument which is, in principle, formalizable in ZFC).

3. Find the mistake in the argument below.

   > **Theorem.** ZFC *is inconsistent.*
   >
   > *Proof.* Let $\{\theta_n : n < \omega\}$ be an enumeration of all formulas of $\mathcal{L}_\in$ with exactly one free variable. Let $\psi(x)$ be the formula "$x \in \omega \wedge \neg\theta_x(x)$". Since ψ is a formula of $\mathcal{L}_\in$ in one free variable, there exists $e \in \omega$ such that $\psi = \theta_e$. But then $\mathsf{ZFC} \vdash \theta_e(e) \leftrightarrow \psi(e) \leftrightarrow \neg\theta_e(e)$. □

Part B: relativization and relative consistency

1. (a) Recall that Δ_0 formulas are absolute for all transitive models of set theory. A formula is called Σ_1 if it has the form $\exists x_0 \dots \exists x_k\, \theta$ for a Δ_0-formula θ, and Π_1 if it has the form $\forall x_0 \dots \forall x_k\, \theta$ for a Δ_0-formula θ. Show that for all transitive models of set theory and all Σ_1-formulas ϕ we have $\phi^M \to \phi$, while for all Π_1-formulas ψ we have: $\psi \to \psi^M$ (we call the former *upwards absoluteness* and the latter *downwards absoluteness*).

   (b) In general, the properties "being a cardinal", "being of the same cardinality" and similar statements are not absolute for transitive models. Show that the statement "$|x| = |y|$" is upwards absolute for transitive models, and the statement "κ is a cardinal" is downwards absolute for transitive models (you may use the fact that "f is a function", "f is a bijection", "α is an ordinal", and the concepts $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ are all Δ_0 and therefore absolute).

2. (a) Let $F : V \to V$ be a bijective class-function. Define $E \subseteq V \times V$ by:

$$xEy \;:\Leftrightarrow\; x \in F(y).$$

   We claim that (V, E) is a model of $\mathsf{ZFC} - \mathsf{Foundation}$. Choose any two axioms of $\mathsf{ZFC} - \mathsf{Foundation}$, and prove that they hold in (V, E).

   (b) Use the previous claim to show

$$\mathrm{Con}(\mathsf{ZFC}) \quad\to\quad \mathrm{Con}(\mathsf{ZFC} - \mathsf{Foundation} + \text{“}\exists x\,(x = \{x\})\text{”})$$

   [Hint: use $F(0) := 1$ and $F(1) := 0$].

Part C: Reflection and elementary submodels

1. Prove the following:

   (a) Let M be an *elementary submodel* of N, i.e., $(M, \in) \preccurlyeq (N, \in)$. Let $c \in N$ be an element which is *uniquely definable in* N; that means that there exists a formula $\phi(x)$ such that
   $$N \models \forall x\, (\phi(x) \leftrightarrow x = c).$$
   Then $c \in M$.

   (b) If $M \preccurlyeq H_{\omega_2}$ then $\omega_1 \in M$.

   (c) If $M \preccurlyeq V_\omega$ then $M = V_\omega$.

   Hint: Prove, by $\in$-induction, that every $x \in V_\omega$ is uniquely definable in V_ω (in the sense of (a)).