

Reflection Theorems Presentation

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1 Introduction and Contents

This presentation will be about the Reflection Theorems. We will cover the following subjects:

1. Elementary Submodels
2. Tarski–Vaught Criterion, Skolem Functions and Skolem Hulls
3. Downward Löwenheim–Skolem Theorem
4. Reflection Principle
5. Mostowski Collapse

2 Elementary Submodels

Definition 2.1 (Submodel). Let $\mathfrak{A}$ be a structure.

A *submodel* of $\mathfrak{A}$ is a subset $B \subseteq A$ endowed with:

- the relations $P^{\mathfrak{A}} \cap B^n$,
- the functions $F^{\mathfrak{A}} \upharpoonright B^m$,
- the constants $c^{\mathfrak{A}}$,

such that:

- every constant $c^{\mathfrak{A}}$ belongs to B ,
- B is closed under all functions $F^{\mathfrak{A}}$; that is,

$$(x_1, \dots, x_m) \in B^m \Rightarrow F^{\mathfrak{A}}(x_1, \dots, x_m) \in B.$$

Definition 2.2 (Elementary Submodel). Let $\mathfrak{B} \subseteq \mathfrak{A}$ be structures in the same language.

We say that $\mathfrak{B}$ is an *elementary submodel* of $\mathfrak{A}$, written

$$\mathfrak{B} \prec \mathfrak{A},$$

if for every formula $\varphi(x_1, \dots, x_n)$ and every $a_1, \dots, a_n \in B$,

$$\mathfrak{B} \models \varphi[a_1, \dots, a_n] \iff \mathfrak{A} \models \varphi[a_1, \dots, a_n].$$

3 Tarski–Vaught Criterion, Skolem Functions and the Skolem Hull

Lemma 3.1 (Tarski–Vaught Criterion). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for a language L with $\mathfrak{A} \subseteq \mathfrak{B}$. Then the following are equivalent:*

1. $\mathfrak{A} \prec \mathfrak{B}$.

2. *For every existential formula $\varphi(\vec{x})$ of L (of the form $\exists y \psi(\vec{x}, y)$) and every $\vec{a} \in A$:
if*

$$\mathfrak{B} \models \varphi(\vec{a}),$$

then there exists $b \in A$ such that

$$\mathfrak{B} \models \psi(\vec{a}, b).$$

The criterion says that elementary submodels must contain *witnesses for existential statements*.

Definition 3.2 (Skolem Function). Let $\mathfrak{A}$ be a structure and $\varphi(x, x_1, \dots, x_n)$ a formula in the language of $\mathfrak{A}$.

A function

$$h : A^n \rightarrow A$$

is called a *Skolem function* for φ if for all $a_1, \dots, a_n \in A$:

$$\text{if } \exists a \in A \mathfrak{A} \models \varphi[a, a_1, \dots, a_n], \text{ then } \mathfrak{A} \models \varphi[h(a_1, \dots, a_n), a_1, \dots, a_n].$$

Definition 3.3. Let $A \subseteq M$.

The *Skolem hull* of A , denoted $\text{Hull}(A)$, is the smallest subset of M containing A and closed under all Skolem functions.

Theorem 3.4. *$\text{Hull}(A)$ is an elementary submodel of M .*

4 Downward Löwenheim–Skolem

Theorem 4.1 (Downward Löwenheim–Skolem–Tarski). *Let $\mathfrak{B}$ be a structure for a language L . Fix a cardinal κ such that*

$$\max(|L|, \aleph_0) \leq \kappa \leq |B|,$$

and let $S \subseteq B$ with $|S| \leq \kappa$.

Then there exists a structure $\mathfrak{A} \prec \mathfrak{B}$ such that

$$S \subseteq A \quad \text{and} \quad |A| = \kappa.$$

Proof. Following the terminology in the Tarski–Vaught lemma, we proceed as follows:

Step 1: Choose Skolem functions.

For each existential formula $\varphi(x, x_1, \dots, x_n)$ of L , define a Skolem function $f_\varphi : B^n \rightarrow B$ as follows:

For $a_1, \dots, a_n \in B$, if there exists $b \in B$ such that

$$\mathfrak{B} \models \varphi[b, a_1, \dots, a_n],$$

then set

$$f_\varphi(a_1, \dots, a_n) = \text{one such } b.$$

Otherwise, choose $f_\varphi(a_1, \dots, a_n)$ arbitrarily.

Such functions can be defined from a well-ordering of B .

Step 2: Collect all Skolem functions.

Let F be the family of all these Skolem functions. Then

$$|F| \leq \kappa$$

because $|L| \leq \kappa$.

Step 3: Extend the set of parameters.

Let $S' \subseteq B$ satisfy $S \subseteq S'$ and $|S'| = \kappa$.

Step 4: Construct the Skolem hull.

Let A be the closure of S' under all functions in F :

$$A = \text{Hull}_F(S').$$

By standard cardinality arguments, $|A| \leq \kappa$, and since $|S'| = \kappa$, we have $|A| = \kappa$.

Step 5: Verify closure under functions of L .

If $g \in L$ is an n -ary function symbol, then $g^\mathfrak{B} : B^n \rightarrow B$. By construction, A is closed under all Skolem functions, so in particular $g^\mathfrak{B}(A^n) \subseteq A$.

For 0-ary constants $c \in L$, note that the corresponding Skolem function maps the empty tuple to c , so $c \in A$.

Step 6: Verify elementary submodel.

By the Tarski–Vaught criterion, $A \prec \mathfrak{B}$, since every existential formula with parameters in A has a witness in A .

Hence $\mathfrak{A} = A$ is an elementary substructure of $\mathfrak{B}$ with

$$S \subseteq A \quad \text{and} \quad |A| = \kappa.$$

□

Remark 4.2 (Skolem Paradox). Since the language of set theory is countable, the Downward Löwenheim–Skolem Theorem implies that if ZFC has a model, then it has a countable model M .

However, ZFC proves the existence of uncountable sets, for example ω_1 . Hence there is some $\beta \in M$ such that

$$M \models \text{“}\beta \text{ is uncountable”}.$$

From the point of view of V , the set M is countable, and therefore every element of M , including β , is contained in a countable set. Thus β is countable.

This situation is known as the *Skolem paradox*. It is not a genuine contradiction, since the notion of “uncountable” is not absolute: M contains no bijection between β and ω , but such a bijection may exist outside of M .

5 Reflection Principle

Theorem 5.1 (Reflection Principle). *Let $\varphi(x_1, \dots, x_n)$ be a formula and M_0 any set. Then:*

1. There exists a set $M \supseteq M_0$ such that for all $x_1, \dots, x_n \in M$,

$$\varphi^M(x_1, \dots, x_n) \iff \varphi(x_1, \dots, x_n).$$

We say that M reflects φ .

2. Moreover, there exists a transitive set $M \supseteq M_0$ that reflects φ . In fact, there exists a limit ordinal α such that

$$M_0 \subseteq V_\alpha \quad \text{and} \quad V_\alpha \text{ reflects } \varphi.$$

3. Assuming the Axiom of Choice, there exists $M \supseteq M_0$ that reflects φ with

$$|M| \leq |M_0| \cdot \aleph_0.$$

In particular, there exists a countable M that reflects φ .

Lemma 5.2.

1. Let $\varphi(u_1, \dots, u_n, x)$ be a formula. For each set M_0 , there exists a set $M \supseteq M_0$ such that for all $u_1, \dots, u_n \in M$,

$$\text{if } \exists x \varphi(u_1, \dots, u_n, x) \text{ in } V, \text{ then } \exists x \in M \varphi(u_1, \dots, u_n, x).$$

Assuming the Axiom of Choice, there exists $M' \supseteq M_0$ satisfying the above and

$$|M'| \leq |M_0| \cdot \aleph_0.$$

2. If $\varphi_1, \dots, \varphi_k$ are formulas, then for each M_0 there exists $M \supseteq M_0$ such that the above property holds simultaneously for each φ_i , $i = 1, \dots, k$.

Proof. (proof of part (i))

Step 1: Define the Skolem-like operation.

For each tuple $(u_1, \dots, u_n)$, define

$$C = \{x : \varphi(u_1, \dots, u_n, x)\},$$

and let

$$\widehat{C} = \{x \in C : \forall z \in C, \text{rank}(x) \leq \text{rank}(z)\}.$$

Set

$$H(u_1, \dots, u_n) = \widehat{C}.$$

By construction, $H(u_1, \dots, u_n)$ has the property that

$$\text{if } \exists x \varphi(u_1, \dots, u_n, x) \text{ in } V, \text{ then } \exists x \in H(u_1, \dots, u_n) \text{ such that } \varphi(u_1, \dots, u_n, x).$$

Step 2: Build the set M (the Skolem Hull) by induction.

Let

$$M_0 \text{ be given, } M_{i+1} = M_i \cup \bigcup \{H(u_1, \dots, u_n) : u_1, \dots, u_n \in M_i\},$$

and define

$$M = \bigcup_{i=0}^{\infty} M_i.$$

Then for any $u_1, \dots, u_n \in M$, there exists i such that $u_1, \dots, u_n \in M_i$, and if $\varphi(u_1, \dots, u_n, x)$ holds for some x , then by construction it holds for some $x \in M_{i+1} \subseteq M$.

Step 3: Using the Axiom of Choice for cardinality bounds.

Assume AC. Let F be a choice function on $\mathcal{P}(M)$. For each tuple $u_1, \dots, u_n \in M$, define

$$h(u_1, \dots, u_n) = F(H(u_1, \dots, u_n)),$$

undefined if $H(u_1, \dots, u_n) = \emptyset$.

Define inductively

$$M'_0 = M_0, \quad M'_{i+1} = M'_i \cup \{h(u_1, \dots, u_n) : u_1, \dots, u_n \in M'_i\},$$

and set

$$M' = \bigcup_{i=0}^{\infty} M'_i.$$

Step 4: Verification.

By construction, M' satisfies condition (12.23):

if $\exists x \varphi(u_1, \dots, u_n, x)$ in V , then $\exists x \in M' \varphi(u_1, \dots, u_n, x)$ for all $u_1, \dots, u_n \in M'$.

Moreover, since each M'_i has cardinality at most $|M_0| \cdot \aleph_0$, it follows that

$$|M'| \leq |M_0| \cdot \aleph_0.$$

□

Proof of the Reflection Principle (Theorem 12.14). Let $\varphi(x_1, \dots, x_n)$ be a formula.

Step 1: Reduce to existential formulas.

We may assume that universal quantifiers do not occur in φ , since $\forall x \psi(x)$ can be rewritten as $\neg \exists x \neg \psi(x)$.

Let $\varphi_1, \dots, \varphi_k$ be all subformulas of φ .

Step 2: Apply the essential lemma.

Given a set M_0 , by Lemma 5.2(ii) there exists a set $M \supseteq M_0$ such that for all $u_1, \dots, u_m \in M$ and for each $j = 1, \dots, k$,

$$\exists x \varphi_j(u_1, \dots, u_m, x) \implies \exists x \in M \varphi_j(u_1, \dots, u_m, x).$$

Step 3: Induction on formula complexity.

We claim that M reflects each φ_j , and in particular φ . The proof is by induction on the complexity of φ_j :

- Atomic formulas: trivially reflected.
- Boolean connectives: if M reflects ψ and χ , then M reflects $\neg\psi$, $\psi \wedge \chi$, $\psi \vee \chi$, $\psi \rightarrow \chi$, and $\psi \leftrightarrow \chi$.
- Existential formulas: suppose M reflects $\varphi_j(u_1, \dots, u_m, x)$. Then for $u_1, \dots, u_m \in M$:

$$\begin{aligned} M \models \exists x \varphi_j(u_1, \dots, u_m, x) &\iff \exists x \in M M \models \varphi_j(u_1, \dots, u_m, x) \\ &\iff \exists x \in M \varphi_j(u_1, \dots, u_m, x) \iff \exists x \varphi_j(u_1, \dots, u_m, x), \end{aligned}$$

where the last equivalence follows from the lemma construction.

This proves part (i) of the theorem.

Step 4: Cardinality bound.

Part (iii) follows by choosing M of size at most $|M_0| \cdot \aleph_0$, as in the lemma.

Step 5: Transitive / V_α version.

To prove part (ii), modify the construction of M so that the sets M_i in the inductive construction are replaced by their *transitive closure*, or by the least $V_\gamma \supseteq M_{i+1}$. Then M is transitive, or $M = V_\alpha$ for a limit ordinal α . □

6 Mostowski Collapse

Definition 6.1 (Extension of an element under a relation). Let E be a binary relation on a class P . For each $x \in P$, define

$$\text{ext}_E(x) = \{z \in P : z E x\},$$

called the *extension of x under E* .

Definition 6.2 (Well-Founded Relation). A relation E on a class P is called *well-founded* if:

1. Every nonempty set $x \subseteq P$ has an E -minimal element, i.e., there exists $m \in x$ such that there is no $z \in x$ with $z E m$.
2. For every $x \in P$, the extension $\text{ext}_E(x)$ is a set.

Definition 6.3 (Extensional Well-Founded Relation). A well-founded relation E on a class P is called *extensional* if for all distinct elements $X, Y \in P$,

$$\text{ext}_E(X) \neq \text{ext}_E(Y),$$

Theorem 6.4 (Mostowski's Collapsing Theorem). *Let E be a well-founded and extensional relation on a class P . Then:*

1. *There exists a transitive class M and an isomorphism*

$$\pi : (P, E) \longrightarrow (M, \in).$$

Moreover, M and π are unique.

2. *In particular, every extensional class P is isomorphic to a transitive class M . Again, M and π are unique.*
3. *In the situation of (ii), if $T \subseteq P$ is transitive (as a class), then*

$$\pi(x) = x \quad \text{for every } x \in T.$$

proof of (2). **Step 1: Definition of the collapse map.**

Since E is well-founded, we may define a function π by well-founded induction. Thus for each $x \in P$ we define

$$\pi(x) = \{\pi(z) : z E x\}. \tag{6.5}$$

In the special case $E = \in$, this becomes

$$\pi(x) = \{\pi(z) : z \in x \cap P\}. \quad (6.6)$$

Let $M = \pi(P)$. By definition (6.5), every element of $\pi(x)$ is of the form $\pi(z)$ for some zEx , hence $\pi(z) \in M$. Therefore M is a transitive class.

Step 2: Injectivity of π .

We use the extensionality of E to show that π is one-to-one. Suppose otherwise. Let $z \in M$ be of minimal rank such that

$$z = \pi(x) = \pi(y)$$

for some $x \neq y$.

Since E is extensional, we have

$$\text{ext}_E(x) \neq \text{ext}_E(y).$$

Thus there exists $u \in \text{ext}_E(x)$ with $u \notin \text{ext}_E(y)$. Let

$$t = \pi(u).$$

Since uEx , we have $t \in \pi(x) = z$. But also $z = \pi(y)$, so there exists $v \in \text{ext}_E(y)$ such that

$$t = \pi(v).$$

Thus

$$t = \pi(u) = \pi(v)$$

with $u \neq v$, and since $t \in z$, the rank of t is smaller than the rank of z . This contradicts the minimal choice of z . Hence π is injective.

Step 3: Preservation of the relation.

We now show

$$xEy \iff \pi(x) \in \pi(y).$$

($\Rightarrow$)

If xEy , then by definition (6.5) we immediately have $\pi(x) \in \pi(y)$.

($\Leftarrow$)

Conversely, if $\pi(x) \in \pi(y)$, then by (6.5) there exists zEy such that

$$\pi(x) = \pi(z).$$

Since π is injective, it follows that $x = z$, hence xEy .

Thus π is an isomorphism between (P, E) and $(M, \in)$.

Step 4: Uniqueness.

Suppose $\pi_1 : (P, E) \rightarrow (M_1, \in)$ and $\pi_2 : (P, E) \rightarrow (M_2, \in)$ are two such isomorphisms. Then

$$\pi_2 \circ \pi_1^{-1}$$

is an isomorphism between $(M_1, \in)$ and $(M_2, \in)$. Where this map is the identity. Therefore

$$\pi_1 = \pi_2$$

and consequently $M_1 = M_2$.

Step 5: Proof of (3).

Let $T \subseteq P$ be transitive. Then for every $x \in T$ we have $x \subseteq P$, and therefore $x \cap P = x$. Hence by (6.5)

$$\pi(x) = \{\pi(z) : z \in x\}.$$

Using $\in$ -induction on x , it follows that

$$\pi(x) = x$$

for all $x \in T$. □