Classical results: Assume the Axiom of Determinacy (AD). Then all sets of reals are Lebesgue Measurable, have the Baire property and the Perfect Set Property.
Classical results: Assume the **Axiom of Determinacy** (AD). Then all sets of reals are **Lebesgue Measurable**, have the **Baire property** and the **Perfect Set Property**.

**Example:** AD $\rightarrow$ Baire property.

**Proof:**
- Define the Banach-Mazur game, $G^{**}$.
- Show: $I$ wins $G^{**}(A) \iff A$ is comeager in an open set, $II$ wins $G^{**}(A) \iff A$ is meager.
- If all sets satisfy that disjunction, then all sets have the **Baire property**.
Determinacy without AD

AD contradicts AC. Suppose, instead, that we have ZFC: what is determinacy?

\[ \mathcal{D} := \{ A \mid G(A) \text{ is determined} \} \]
AD contradicts AC. Suppose, instead, that we have ZFC: what is determinacy?

$$D := \{ A \mid G(A) \text{ is determined} \}$$

Is determinacy a “mother regularity property”, i.e., does it imply all the other regularity properties?

Answer: it does classwise but not necessarily pointwise.
Let $\Gamma$ be a boldface pointclass (closed under continuous pre-images, and in some cases under intersections with basic open sets).

If all sets in $\Gamma$ are determined then all sets in $\Gamma$ are regular.
Let \( \Gamma \) be a boldface pointclass (closed under continuous pre-images, and in some cases under intersections with basic open sets).

If all sets in \( \Gamma \) are determined then all sets in \( \Gamma \) are regular.

**Definition:** Let \( \text{Reg} \) be some regularity property. Then we say:

- “Determinacy implies \( \text{Reg} \) classwise” \( \iff \) for all boldface pointclasses \( \Gamma \) (\( \Gamma \subseteq D \rightarrow \Gamma \subseteq \text{Reg} \)).
- “Determinacy implies \( \text{Reg} \) pointwise” \( \iff \) \( D \subseteq \text{Reg} \).
Example: Let $\Gamma$ be a boldface pointclass. If $\Gamma \subseteq D$ then $\Gamma \subseteq \text{BP}$.

Proof:

- Define the Banach-Mazur game, $G^{**}$.
- Encode $A \mapsto A'$ so that $G^{**}(A) \equiv G(A')$.

Then: $I$ wins $G(A') \iff A$ is comeager in an open set $II$ wins $G(A') \iff A$ is meager.

If $A \in \Gamma$ then $A' \in \Gamma$ so $G(A')$ is determined. Then $A$ is either comeager in an open set or meager.

If all sets in $\Gamma$ have this property, then all sets in $\Gamma$ have the Baire property. \qed
Pointwise “mother regularity property” = e.g. homogeneously Suslin sets, and not determinacy.

Pointwise "mother regularity property" = e.g. homogeneously Suslin sets, and **not** determinacy.


Sets can be **determined** but not **regular** (AC).

**My MSc thesis:** continue this investigation.
Arboreal Forcings

Definition:

A forcing partial order \((\mathbb{P}, \leq)\) is called **arboreal** if it is isomorphic to a collection \(\mathcal{T}\) of perfect **trees** on \(\omega\) or \(2\) ordered by inclusion, with the extra condition that

\[
\forall T \in \mathcal{T} \ \forall t \in T \ \exists S \in \mathcal{T} \ (S \subseteq T \ \wedge \ t \subseteq \text{stem}(S))
\]

An arboreal \((\mathbb{P}, \leq)\) is called **topological** if \(\{[P] \mid P \in \mathbb{P}\}\) is a topology base on \(\omega^\omega\) or \(2^\omega\). Otherwise, it is called **non-topological**.
Examples

Some examples: (non-topological)

- Sacks forcing $S$: all perfect trees.

- Miller forcing $M$: all super-perfect trees.

- Laver forcing $L$: all trees with finite stem and afterwards $\omega$-splitting.
Examples (2)

Some examples: (topological)

- Cohen forcing $\mathbb{C}$: basic open sets $[s]$.

- Hechler forcing $\mathbb{D}$: for $s \in \omega^{<\omega}$ and $f \in \omega^\omega$ with $s \subseteq f$, define $[s, f] := \{x \in \omega^\omega \mid s \subseteq x \land \forall n \geq |s| (x(n) \geq f(n))\}$. 
Regularity Properties

Definition: Given \((\mathbb{P}, \leq)\), we define the **Marczewski-Burstin** algebra of \(\mathbb{P}\):

\[ A \in \text{MB}(\mathbb{P}) \iff \forall P \in \mathbb{P} \exists Q \leq P ([Q] \subseteq A \lor [Q] \cap A = \emptyset) \]
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- the **weak Marczewski-Burstin** algebra of \(\mathbb{P}\):

  \[ A \in \text{wMB}(\mathbb{P}) \iff \exists Q \in \mathbb{P} ([Q] \subseteq A \lor [Q] \cap A = \emptyset) \]
Definition: Given \((\mathcal{P}, \leq)\), we define

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  \]

- the **weak Marczewski-Burstin** algebra of \(\mathcal{P}\):
  \[
  A \in \text{wMB}(\mathcal{P}) : \iff \exists Q \in \mathcal{P} \ ([Q] \subseteq A \lor [Q] \cap A = \emptyset)
  \]

- If \(\mathcal{P}\) is **topological**, then we define
  \[
  \text{BP}(\mathcal{P}) := \{ A \mid A \text{ has the Baire property in } (\omega^\omega, \mathcal{P}) \} 
  \]
Pointwise View of Determinacy

In “The pointwise view of determinacy”, the following results were proved: For non-topological $\mathbb{P}$:

1. $\mathbb{D} \not\rightarrow \text{MB}(\mathbb{P})$ pointwise (i.e., $\mathbb{D} \not\subset \text{MB}(\mathbb{P})$).

2. $\mathbb{P}$ classified into three cases:
   - Case 1: $\mathbb{D} \rightarrow \text{wMB}(\mathbb{P})$ pointwise.
   - Case 2: $\mathbb{D} \not\rightarrow \text{wMB}(\mathbb{P})$ pointwise.
   - Case 3: There are examples either way.
In “The pointwise view of determinacy”, the following results were proved: For non-topological \( \mathbb{P} \):

1. \( \mathcal{D} \not\rightarrow \text{MB}(\mathbb{P}) \) pointwise (i.e., \( \mathcal{D} \not\subseteq \text{MB}(\mathbb{P}) \)).

2. \( \mathbb{P} \) classified into three cases:
   - Case 1: \( \mathcal{D} \rightarrow \text{wMB}(\mathbb{P}) \) pointwise.
   - Case 2: \( \mathcal{D} \not\rightarrow \text{wMB}(\mathbb{P}) \) pointwise.
   - Case 3: There are examples either way.

**Question:** Can the same analysis be done for topological \( \mathbb{P} \) and \( \text{BP}(\mathbb{P}) \)? What about \( \text{wBP}(\mathbb{P}) \)?
Theorem: If $\mathcal{P}$ is non-atomic then $D \not	o BP(\mathcal{P})$ pointwise.
Theorem: If $\mathbb{P}$ is non-atomic then $D \not\rightarrow \text{BP}(\mathbb{P})$ pointwise.

Proof:

If $A \in \text{BP}(\mathbb{P})$ then for every open $O$ there is a perfect tree $T$ in $O$ such that $[T] \subseteq A$ or $[T] \cap A = \emptyset$. 
**Baire Property**

**Theorem:** If $\mathbb{P}$ is non-atomic then $D \not\to \text{BP}(\mathbb{P})$ pointwise.

**Proof:**

1. If $A \in \text{BP}(\mathbb{P})$ then for every open $O$ there is a perfect tree $T$ in $O$ such that $[T] \subseteq A$ or $[T] \cap A = \emptyset$.

2. Find a $P \in \mathbb{P}$ and a strategy $\sigma$ such that $[P] \cap [\sigma] = \emptyset$. 

**Baire Property**

**Theorem:** If $\mathbb{P}$ is non-atomic then $D \notightarrow \text{BP}(\mathbb{P})$ pointwise.

**Proof:**

- If $A \in \text{BP}(\mathbb{P})$ then for every open $O$ there is a perfect tree $T$ in $O$ such that $[T] \subseteq A$ or $[T] \cap A = \emptyset$.
- Find a $P \in \mathbb{P}$ and a strategy $\sigma$ such that $[P] \cap [\sigma] = \emptyset$.
- Let $\langle T_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ enumerate all perfect trees in $[P]$. 

![Diagram](image-url)
Baire Property

**Theorem:** If \( \mathcal{P} \) is non-atomic then \( \text{D} \not\rightarrow \text{BP}(\mathcal{P}) \) pointwise.

**Proof:**

- If \( A \in \text{BP}(\mathcal{P}) \) then for every open \( O \) there is a perfect tree \( T \) in \( O \) such that \([T] \subseteq A \) or \([T] \cap A = \emptyset\).
- Find a \( P \in \mathcal{P} \) and a strategy \( \sigma \) such that \([P] \cap [\sigma] = \emptyset\).
- Let \( \langle T_\alpha \mid \alpha < 2^{\aleph_0} \rangle \) enumerate all perfect trees in \([P]\).
- Since also \( |T_\alpha| = 2^{\aleph_0} \), we find two Bernstein components \( A \) and \( B \) with \( A \cap B = \emptyset \) and

\[
\forall \alpha < 2^{\aleph_0} \ (A \cap [T_\alpha] \neq \emptyset \land B \cap [T_\alpha] \neq \emptyset)
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**Baire Property**

**Theorem:** If $\mathbb{P}$ is **non-atomic** then $D \not	o \text{BP}(\mathbb{P})$ pointwise.

**Proof:**

- If $A \in \text{BP}(\mathbb{P})$ then for every open $O$ there is a perfect tree $T$ in $O$ such that $[T] \subseteq A$ or $[T] \cap A = \emptyset$.

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\[
\forall \alpha < 2^{\aleph_0} \ (A \cap [T_\alpha] \neq \emptyset \land B \cap [T_\alpha] \neq \emptyset)
\]

- Let $A' := A \cup [\sigma]$. Then for **no** perfect tree $T$ in $[P]$ do we have $[T] \subseteq A'$ or $[T] \cap A' = \emptyset$, so neither $A'$ nor its complement is in $\text{BP}(\mathbb{P})$. But either $A'$ or its complement is determined. $\Box$
Weak Baire property

How to define $wBP(P)$?
Weak Baire property

How to define \( wBP(\mathbb{P}) \)?

Use the following fact: \( A \) has the Baire property iff

\[
\forall O \ \exists U \subseteq O \ (U \cap A \text{ is meager } \lor \ U \setminus A \text{ is meager})
\]
Weak Baire property

How to define \( wBP(\mathcal{P}) \)?

Use the following fact: \( A \) has the Baire property iff

\[
\forall O \, \exists U \subseteq O \ (U \cap A \text{ is meager} \lor U \setminus A \text{ is meager})
\]

Definition: \( A \) has the weak Baire property iff

\[
\exists U \ (U \cap A \text{ is meager} \lor U \setminus A \text{ is meager})
\]
Three Cases

Consider the topological space $(\omega^\omega, P)$ or $(2^\omega, P)$.

Case 1: For every $\sigma$:
\[ \exists P \in P \text{ s.t. } [P] \setminus [\sigma] \text{ is meager} \]

Case 2: For some $\sigma$:
\[ \forall P \in P \exists Q \leq P \text{ s.t. } [Q] \cap [\sigma] \text{ is meager} \]

Case 3: None of the above.
Consider the topological space $(\omega^\omega, \mathcal{P})$ or $(2^\omega, \mathcal{P})$.

**Case 1:** For every $\sigma$:

$$\exists P \in \mathcal{P} \text{ s.t. } [P] \setminus [\sigma] \text{ is meager}$$

**Case 2:** For some $\sigma$:

$$\forall P \in \mathcal{P} \exists Q \leq P \text{ s.t. } [Q] \cap [\sigma] \text{ is meager}$$

**Case 3:** None of the above.

**Case 1:** $D \rightarrow \text{wBP}(\mathcal{P})$ pointwise.

**Case 2:** $D \nrightarrow \text{wBP}(\mathcal{P})$ pointwise.

**Case 3:** ?
Three Cases

Consider the topological space \((\omega^\omega, P)\) or \((2^{\omega}, P)\).

**Case 1:** For every \(\sigma\):
\[\exists P \in P \text{ s.t. } [P] \setminus [\sigma] \text{ is meager}\]

**Case 2:** For some \(\sigma\):
\[\forall P \in P \exists Q \leq P \text{ s.t. } [Q] \cap [\sigma] \text{ is meager}\]

**Case 3:** None of the above.

**Case 1:** \(D \rightarrow w\text{BP}(P)\) pointwise.

**Case 2:** \(D \not\rightarrow w\text{BP}(P)\) pointwise. 

**Case 3:** ?

All standard \(P\) belong to this category.

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Asymmetric Properties

Perfect Set Property: $A$ is countable or contains a perfect tree. This is an asymmetric property, saying that “$A$ is big or small”.

PSP = “Asym$(S)$” (Sacks forcing).

Analogous asymmetric properties have been defined for other $(\mathbb{P}, \leq)$.
Asymmetric Properties

Perfect Set Property: \( A \) is countable or contains a perfect tree. This is an asymmetric property, saying that “\( A \) is big or small”.

\[
PSP = "\text{Asym}(S)" \text{ (Sacks forcing)}.\]

Analogous asymmetric properties have been defined for other \((\mathbb{P}, \leq)\).

Question:

1. Is there a general definition for \( \text{Asym}(\mathbb{P}) \)?
2. What about \( D \rightarrow \text{Asym}(\mathbb{P}) \) pointwise?
Notions of Smallness

Definition:

- For two reals $x, y$, define
  \[ x \leq^{*} y :\iff \forall n \in \mathbb{N} (x(n) \leq y(n)) \]

- $A \subseteq \omega^\omega$ is **σ-bounded** iff $\exists f \forall x \in A (x \leq^{*} f)$.

- $A \subseteq \omega^\omega$ is **dominating** iff $\forall f \exists x \in A (f \leq^{*} x)$.

- $A \subseteq \omega^\omega$ is **strongly dominating** iff
  \[ \forall f \exists x \in A \forall n (x(n+1) > f(x(n))) \]
Notions of Largeness

**Definition:**

- **Perfect tree:** every node has an extension which is a splitting node.

- **Super-perfect tree:** every splitting node is $\omega$-splitting and every node has an extension which is an $\omega$-splitting node.

- **Spinas tree:** super-perfect tree such that for every $t \in T$:
  \[ \forall s_1, s_2 \ (t \sim s_1 \text{ and } t \sim s_2 \text{ are } \omega\text{-splitting nodes of } T \rightarrow |s_1| = |s_2|) \]
  i.e., the next splitting node is a fixed distance away from $t$.

- **Laver tree:** the stem is finite, and after the stem, every node is $\omega$-splitting.
## Asymmetric properties

<table>
<thead>
<tr>
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<td>$L^*$</td>
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</tr>
</tbody>
</table>

What do all these properties have in common?
Asymmetric games

Definition:

1. Asymmetric Game $G^\bullet(A)$:

   $I: \quad s_0 \quad s_1 \quad \ldots$

   $II: \quad n_1 \quad n_2 \quad \ldots$

   $s_i \in \omega^\omega \setminus \{\emptyset\}, \quad n_i \in \omega$

   $I$ wins iff

   \begin{itemize}
   \item $\forall i \geq 1: s_i(0) \neq n_i$
   \item $x := s_0 \prec s_1 \prec s_2 \prec \cdots \in A$.
   \end{itemize}
Asymmetric games

Definition:

1. Asymmetric Game $G^*(A)$:

\[
I : \quad s_0 \quad s_1 \quad \ldots \\
II : \quad n_1 \quad n_2 \quad \ldots
\]

$I$ wins iff

\[\forall i \geq 1 : s_i(0) \neq n_i\]
\[x := s_0 \triangledown s_1 \triangledown s_2 \triangledown \cdots \in A.\]

$s_i \in \omega^\omega \setminus \{\emptyset\}, \quad n_i \in \omega$

2. Kechris Game $\check{G}(A)$:

\[
I : \quad s_0 \quad s_1 \quad \ldots \\
II : \quad n_1 \quad n_2 \quad \ldots
\]

$I$ wins iff

\[\forall i \geq 1 : s_i(0) \geq n_i\]
\[x := s_0 \triangledown s_1 \triangledown s_2 \triangledown \cdots \in A.\]

$s_i \in \omega^\omega \setminus \{\emptyset\}, \quad n_i \in \omega$
Asymmetric games

Definition:

3. **Spinas Game** \( G_u(A) \):

\[
\begin{array}{ccc}
I : & (s_0, k_0) & (s_1, k_1) & \ldots \\
II : & n_1 & n_2 \\
\end{array}
\]

\( s_i \in \omega^\omega \setminus \{\emptyset\}, \ k_i \in \omega \setminus \{0\}, \ n_i \in \omega \)

\( I \) wins iff

- \( \forall i \geq 1 : |s_i| = k_i - 1 \)
- \( \forall i \geq 1 : s_i(0) \geq n_i \)
- \( x := s_0 \cdot s_1 \cdot s_2 \cdot \ldots \in A. \)
Asymmetric games

Definition:

3. **Spinasis Game** $G_u(A)$:

$I$: 

\[
(s_0, k_0) \quad (s_1, k_1) \quad \ldots
\]

$II$: 

\[
n_1 \quad n_2
\]

$s_i \in \omega^\omega \setminus \{\emptyset\}$, $k_i \in \omega \setminus \{0\}$, $n_i \in \omega$

$I$ wins iff

- $\forall i \geq 1: |s_i| = k_i - 1$
- $\forall i \geq 1: s_i(0) \geq n_i$
- $x := s_0 \bar{s}_1 \bar{s}_2 \bar{s}_3 \cdots \in A.$

4. **Goldstern Game** $D(A)$:

$I$: 

\[
s_0 \quad k_1 \quad k_2
\]

$II$: 

\[
n_1 \quad n_2 \quad \ldots
\]

$s_i \in \omega^\omega \setminus \{\emptyset\}$, $k_i \in \omega$, $n_i \in \omega$

$I$ wins iff

- $\forall i \geq 1: k_i > n_i$
- $x := s_0 \bar{\langle} k_1, k_2, \ldots \bar{\rangle} \in A.$
Theorem: (Davis 1964; Kechris 1977; Spinas 1993; Goldstern et al 1995)

1. (a) \( I \) wins \( G^\bullet(A) \iff A \) contains a perfect tree.
   (b) \( II \) wins \( G^\bullet(A) \iff A \) is countable.

2. (a) \( I \) wins \( \tilde{G}(A) \iff A \) contains a super-perfect tree.
   (b) \( II \) wins \( \tilde{G}(A) \iff A \) is \( \sigma \)-bounded.

3. (a) \( I \) wins \( G_u(A) \iff A \) contains a Spinas tree.
   (b) \( II \) wins \( G_u(A) \iff A \) is not dominating.

4. (a) \( I \) wins \( D(A) \iff A \) contains a Laver tree.
   (b) \( II \) wins \( D(A) \iff A \) is not strongly dominating.
**Generalyzed Asymmetric Games**

**Definition:** Start with set of parameters $\Phi := (R, r^0, \{\Theta_n\}_{n \in \omega}, f)$ where

- $R \subseteq \mathcal{P}(\omega^\omega)$ is a countable set of requirements.
- $r^0 \subseteq \omega^\omega$ is the initial requirement.
- The $\Theta_i$ are countable sets of additional information.
- $f : \bigcup_n \Theta_n \rightarrow \mathcal{P}(R)$.

Then the game $G_\Phi(A)$ is defined as follows:

$I$ wins $G_\Phi(A)$ iff

- $s_0 \in r^0$
- $\forall i \geq 1 : s_i \in r_i$
- $x := s_0 \leftarrow s_1 \leftarrow s_2 \leftarrow \cdots \in A$. 

where $s_i \in \omega^\omega \setminus \{\emptyset\}, \theta_i \in \Theta_i, r_i \in R \cap f(\theta_{i-1})$
Definition: Let $(\mathbb{P}, \leq)$ be a forcing notion, and let $G_\Phi$ be a generalized asymmetric game. We say that $G_\Phi$ represents $\mathbb{P}$ iff

$$\forall A \ [I \text{ has a winning strategy in } G_\Phi(A) \iff \exists P \in \mathbb{P} ([P] \subseteq A)]$$
**Definition:** Let $(\mathbb{P}, \leq)$ be a forcing notion, and let $G_{\Phi}$ be a generalized asymmetric game. We say that $G_{\Phi}$ represents $\mathbb{P}$ iff

$$\forall A \ [I \text{ has a winning strategy in } G_{\Phi}(A) \iff \exists P \in \mathbb{P} ([P] \subseteq A)]$$

**Definition:** Suppose $G_{\Phi}$ represents $\mathbb{P}$. Then we define $\text{Asym}_{\Phi}(\mathbb{P})$ by

$$A \in \text{Asym}_{\Phi}(\mathbb{P}) : \iff \exists P \in \mathbb{P} ([P] \subseteq A) \lor \text{II wins } G_{\Phi}(A)$$
Asymmetric Game Characterizations

**Definition:** Let \((\mathbb{P}, \leq)\) be a forcing notion, and let \(G_\Phi\) be a generalized asymmetric game. We say that \(G_\Phi\) **represents** \(\mathbb{P}\) iff

\[
\forall A \ [I \text{ has a winning strategy in } G_\Phi(A) \iff \exists P \in \mathbb{P} ([P] \subseteq A)]
\]

**Definition:** Suppose \(G_\Phi\) represents \(\mathbb{P}\). Then we define \(\text{Asym}_\Phi(\mathbb{P})\) by

\[
A \in \text{Asym}_\Phi(\mathbb{P}) : \iff \exists P \in \mathbb{P} ([P] \subseteq A) \lor II \text{ wins } G_\Phi(A)
\]

**Examples:**

- Let \(\Phi_\bullet\) be the parameters corresponding to Davis’s asymmetric game \(G_\bullet(A)\). Then Sacks forcing is represented by \(G_{\Phi_\bullet}(A)\). Therefore we can write \(\text{PSP} = \text{Asym}_{\Phi_\bullet}(\mathbb{S})\).

- Let \(\Phi_\sim\) be the parameters corresponding to the Kechris game \(\tilde{G}(A)\). Then Miller forcing is represented by \(G_{\Phi_\sim}(A)\). Therefore \(K_\sigma\)-regularity = \(\text{Asym}_{\Phi_\sim}(\mathbb{M})\).
Questions

Does every $\mathbb{P}$ have an asymmetric game $G_\Phi$ which represents it?

If so, is the representation unique?
Questions

Does every $\mathbb{P}$ have an asymmetric game $G_{\Phi}$ which represents it?

If so, is the representation unique?

For example, Silver forcing probably doesn’t have a game representation.

Silver forcing: uniform trees. Perfect trees $T$ on $2^\omega$ such that

$$\forall s, t \in T \ (|s| = |t| \rightarrow \{i \mid s \upharpoonright i \in T\} = \{i \mid t \upharpoonright i \in T\})$$
Pointwise Non-implication

Recall the question: “$D \rightarrow \text{Asym}(P)$ pointwise?”
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**Theorem:** Suppose $P$ is represented by $G_\Phi$ which is non-trivial in the sense that $I$ may choose between $s$ and $t$ with $s \perp t$. Then $D \not\rightarrow \text{Asym}_\Phi(P)$ pointwise.
Pointwise Non-implication

Recall the question: “$D \rightarrow \text{Asym}(P)$ pointwise?”

**Theorem:** Suppose $P$ is represented by $G_\Phi$ which is non-trivial in the sense that $I$ may choose between $s$ and $t$ with $s \perp t$. Then $D \not\rightarrow \text{Asym}_\Phi(P)$ pointwise.

**Proof:**
Recall the question: “$D \rightarrow \text{Asym}(\mathbb{P})$ pointwise?”

**Theorem:** Suppose $\mathbb{P}$ is represented by $G_\Phi$ which is non-trivial in the sense that $I$ may choose between $s$ and $t$ with $s \perp t$. Then $D \not\rightarrow \text{Asym}_\Phi(\mathbb{P})$ pointwise.

**Proof:**

- Fix $s \in \omega^{<\omega}$ which “$I$ may play in the first move”. Fix a $\sigma$ with $s \notin \sigma$. 

Regularity Properties and Determinacy – p.25/27
Recall the question: “$D \to \text{Asym}(P)$ pointwise?”

**Theorem:** Suppose $P$ is represented by $G_\Phi$ which is non-trivial in the sense that $I$ may choose between $s$ and $t$ with $s \perp t$. Then $D \nleftrightarrow \text{Asym}_\Phi(P)$ pointwise.

**Proof:**

- Fix $s \in \omega^{<\omega}$ which “$I$ may play in the first move”. Fix a $\sigma$ with $s \notin \sigma$.

- Let $\langle T_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ enumerate perfect trees and $\langle \tau_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ $II$-strategies. Let $K_\alpha := \{x \mid s \subset x \land x \text{ is a play according to } \tau_\alpha\}$
Pointwise Non-implication

Recall the question: “$D \rightarrow \text{Asym}(P)$ pointwise?”

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- Fix $s \in \omega^{<\omega}$ which “$I$ may play in the first move”. Fix a $\sigma$ with $s \notin \sigma$.
- Let $\langle T_\alpha | \alpha < 2^{\aleph_0} \rangle$ enumerate perfect trees and $\langle \tau_\alpha | \alpha < 2^{\aleph_0} \rangle$ $II$-strategies. Let $K_\alpha := \{x | s \subseteq x \land x$ is a play according to $\tau_\alpha\}$
- By the non-triviality assumption, $|K_\alpha| = 2^{\aleph_0}$ for all $\alpha$. 
Recall the question: “$D \rightarrow \text{Asym}(P)$ pointwise?”

**Theorem:** Suppose $P$ is represented by $G_\Phi$ which is non-trivial in the sense that $I$ may choose between $s$ and $t$ with $s \perp t$. Then $D \not\rightarrow \text{Asym}_\Phi(P)$ pointwise.

**Proof:**

- Fix $s \in \omega^{<\omega}$ which “$I$ may play in the first move”. Fix a $\sigma$ with $s \notin \sigma$.
- Let $\langle T_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ enumerate perfect trees and $\langle \tau_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ II-strategies. Let $K_\alpha := \{x \mid s \subseteq x \land x \text{ is a play according to } \tau_\alpha\}$
- By the non-triviality assumption, $|K_\alpha| = 2^{\aleph_0}$ for all $\alpha$.
- Inductively find Bernstein components $A$ and $B$ as follows: given $A_\alpha$, $B_\alpha$, choose
  
  - $a_{\alpha+1} \in K_\alpha \setminus (A_\alpha \cup B_\alpha)$
  - $b_{\alpha+1} \in \left[ T_\alpha \right] \setminus (A_\alpha \cup B_\alpha \cup \{a_{\alpha+1}\})$

  and let $A_{\alpha+1} := A_\alpha \cup \{a_{\alpha+1}\}$ and $B_{\alpha+1} := B_\alpha \cup \{b_{\alpha+1}\}$. 

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Recall the question: “$D \rightarrow \text{Asym}(\mathbb{P})$ pointwise?”

**Theorem:** Suppose $\mathbb{P}$ is represented by $G_\Phi$ which is non-trivial in the sense that $I$ may choose between $s$ and $t$ with $s \perp t$. Then $D \not\rightarrow \text{Asym}_\Phi(\mathbb{P})$ pointwise.

**Proof:**

- Fix $s \in \omega^{<\omega}$ which “I may play in the first move”. Fix a $\sigma$ with $s \notin \sigma$.
- Let $\langle T_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ enumerate perfect trees and $\langle \tau_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ $II$-strategies. Let $K_\alpha := \{ x \mid s \subseteq x \land x \text{ is a play according to } \tau_\alpha \}$
- By the non-triviality assumption, $|K_\alpha| = 2^{\aleph_0}$ for all $\alpha$.
- Inductively find Bernstein components $A$ and $B$ as follows: given $A_\alpha, B_\alpha$, choose
  - $a_{\alpha+1} \in K_\alpha \setminus (A_\alpha \cup B_\alpha)$
  - $b_{\alpha+1} \in [T_\alpha] \setminus (A_\alpha \cup B_\alpha \cup \{ a_{\alpha+1} \})$
- and let $A_{\alpha+1} := A_\alpha \cup \{ a_{\alpha+1} \}$ and $B_{\alpha+1} := B_\alpha \cup \{ b_{\alpha+1} \}$.
- Then $A \cap [\sigma] = \emptyset$ so $A$ is determined; $A$ doesn’t contain a perfect tree by construction; $II$ doesn’t have a winning strategy in $G_\Phi(A)$ because $\forall \tau_\alpha$, we have $a_{\alpha+1} \in A$ according to $\tau_\alpha$. \qed
Conclusion

There might be a better definition of $\text{Asym} (\mathbb{P})$, but as long as $\text{Asym} (\mathbb{P}) = \text{Asym}_\Phi (\mathbb{P})$ for some non-trivial $G_\Phi$ representing $\mathbb{P}$, we have

**Conclusion:** $D \not
rightarrow \text{Asym} (\mathbb{P})$ pointwise.

In particular $D \not
rightarrow \text{PSP}$, $K_\sigma$-regularity, $u$-regularity and Laver-regularity pointwise.
Thank you!

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