Regularity Properties and Determinacy

Yurii Khomskii

yurii@deds.nl

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

University of Amsterdam

Regularity Properties and Determinacy

Classical results: Assume the **Axiom of Determinacy** (**AD**). Then all sets of reals are Lebesgue Measurable, have the Baire property and the Perfect Set Property.

Regularity Properties and Determinacy

Classical results: Assume the **Axiom of Determinacy** (**AD**). Then all sets of reals are Lebesgue Measurable, have the Baire property and the Perfect Set Property.

Example: AD $\rightarrow$ Baire property.

Proof:

- Define the Banach-Mazur game, G^{**}.
- Show: I wins $G^{**}(A) \;\Leftrightarrow\; A$ is comeager in an open set,
  II wins $G^{**}(A) \;\Leftrightarrow\; A$ is meager.
- If all sets satisfy that disjunction, then all sets have the Baire property. $\square$

Determinacy without AD

AD contradicts **AC**. Suppose, instead, that we have **ZFC**: what is determinacy?

$$\mathbf{D} := \{A \mid G(A) \text{ is determined}\}$$

Determinacy without AD

AD contradicts **AC**. Suppose, instead, that we have **ZFC**: what is determinacy?

$$\mathbf{D} := \{A \mid G(A) \text{ is determined}\}$$

Is determinacy a “mother regularity property”, i.e., does it imply all the other regularity properties?

Answer: it does **classwise** but not necessarily **pointwise**.

Classwise vs. Pointwise

Let Γ be a boldface pointclass

(closed under continuous pre-images, and in some cases under intersections with basic open sets).

If all sets in Γ are determined then all sets in Γ are regular.

Classwise vs. Pointwise

Let Γ be a boldface pointclass

(closed under continuous pre-images, and in some cases under intersections with basic open sets).

If all sets in Γ are determined then all sets in Γ are regular.

Definition: Let $\mathbf{Reg}$ be some regularity property. Then we say:

- "Determinacy implies $\mathbf{Reg}$ classwise" $\iff$ for all boldface pointclasses Γ $(\Gamma \subseteq \mathbf{D} \to \Gamma \subseteq \mathbf{Reg})$.
- "Determinacy implies $\mathbf{Reg}$ pointwise" $\iff$ $\mathbf{D} \subseteq \mathbf{Reg}$

Example

Example: Let $\boldsymbol{\Gamma}$ be a boldface pointclass. If $\boldsymbol{\Gamma} \subseteq$ **D** then $\boldsymbol{\Gamma} \subseteq$ BP.

Proof:

- Define the Banach-Mazur game, G^{**}.
- Encode $A \rightsquigarrow A'$ so that $G^{**}(A) \equiv G(A')$.
- Then: I wins $G(A') \iff A$ is comeager in an open set
  II wins $G(A') \iff A$ is meager.
- If $A \in \boldsymbol{\Gamma}$ then $A' \in \boldsymbol{\Gamma}$ so $G(A')$ is determined. Then A is either comeager in an open set or meager.
- If all sets in $\boldsymbol{\Gamma}$ have this property, then all sets in $\boldsymbol{\Gamma}$ have the Baire property. $\square$

Pointwise

Pointwise "mother regularity property" = e.g. **homogeneously Suslin** sets, and **not** determinacy.

Benedikt Löwe, **The pointwise view of determinacy: arboreal forcings, measurability, and weak measurability**, Rocky Mountains Journal of Mathematics **35** (2005), pp. 1233–1249

Pointwise

Pointwise "mother regularity property" = e.g. **homogeneously Suslin** sets, and **not** determinacy.

Benedikt Löwe, **The pointwise view of determinacy: arboreal forcings, measurability, and weak measurability**, Rocky Mountains Journal of Mathematics **35** (2005), pp. 1233–1249

Sets can be **determined** but not **regular** (**AC**).

My MSc thesis: continue this investigation.

Arboreal Forcings

Definition:

- A **forcing partial order** $(\mathbb{P}, \leq)$ is called **arboreal** if it is isomorphic to a collection $\mathfrak{T}$ of perfect **trees** on ω or 2 ordered by inclusion, with the extra condition that

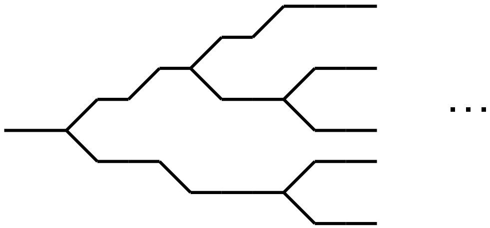
$$\forall T \in \mathfrak{T}\, \forall t \in T\, \exists S \in \mathfrak{T}\, (S \subseteq T \ \wedge \ t \subseteq \text{stem}(S))$$

- An arboreal $(\mathbb{P}, \leq)$ is called **topological** if $\{[P] \mid P \in \mathbb{P}\}$ is a topology base on ω^ω or 2^ω. Otherwise, it is called **non-topological**.

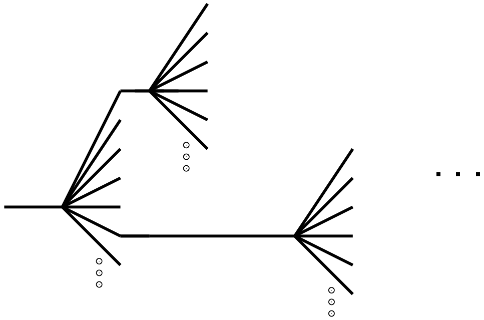
Examples

Some examples: (non-topological)

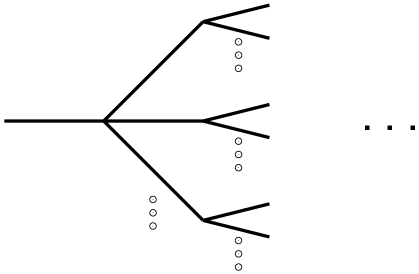
- Sacks forcing $\mathbb{S}$: all perfect trees.



- Miller forcing $\mathbb{M}$: all super-perfect trees.



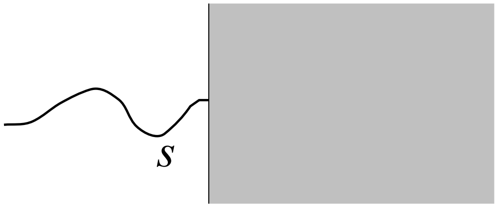
- Laver forcing $\mathbb{L}$: all trees with finite stem and afterwards ω-splitting.



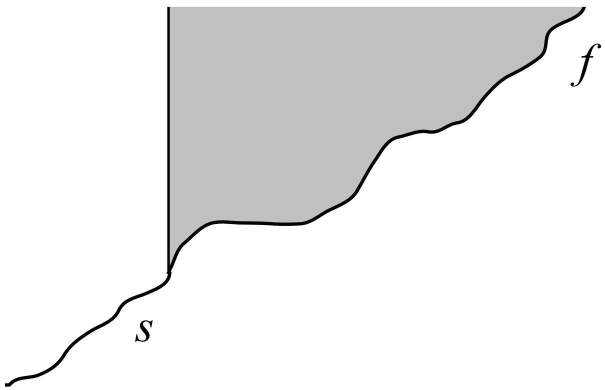
Examples (2)

Some examples: (topological)

- Cohen forcing $\mathbb{C}$: basic open sets $[s]$.



- Hechler forcing $\mathbb{D}$: for $s \in \omega^{<\omega}$ and $f \in \omega^\omega$ with $s \subseteq f$, define $[s, f] := \{x \in \omega^\omega \mid s \subseteq x \wedge \forall n \geq |s|(x(n) \geq f(n))\}$.

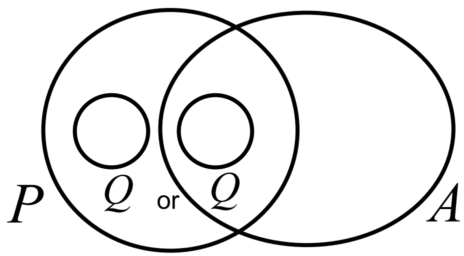


Regularity Properties

Definition: Given $(\mathbb{P}, \leq)$, we define

- the Marczewski-Burstin algebra of $\mathbb{P}$:

$$A \in \mathsf{MB}(\mathbb{P}) \;:\Longleftrightarrow\; \forall P \in \mathbb{P}\, \exists Q \leq P\, ([Q] \subseteq A \;\vee\; [Q] \cap A = \varnothing)$$

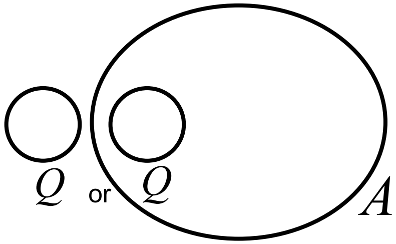


Regularity Properties

Definition: Given $(\mathbb{P}, \leq)$, we define

- the Marczewski-Burstin algebra of $\mathbb{P}$:

$$A \in \mathsf{MB}(\mathbb{P}) \;:\Longleftrightarrow\; \forall P \in \mathbb{P}\, \exists Q \leq P \; ([Q] \subseteq A \;\vee\; [Q] \cap A = \varnothing)$$



- the weak Marczewski-Burstin algebra of $\mathbb{P}$:

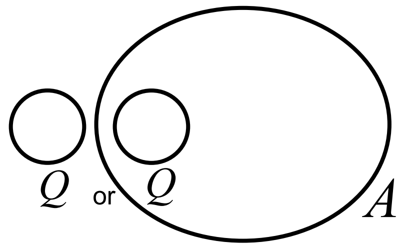
$$A \in \mathsf{wMB}(\mathbb{P}) \;:\Longleftrightarrow\; \exists Q \in \mathbb{P} \; ([Q] \subseteq A \;\vee\; [Q] \cap A = \varnothing)$$

Regularity Properties

Definition: Given $(\mathbb{P}, \leq)$, we define

- the Marczewski-Burstin algebra of $\mathbb{P}$:

$$A \in \mathrm{MB}(\mathbb{P}) \;:\Longleftrightarrow\; \forall P \in \mathbb{P}\, \exists Q \leq P\, ([Q] \subseteq A \;\vee\; [Q] \cap A = \varnothing)$$



- the weak Marczewski-Burstin algebra of $\mathbb{P}$:

$$A \in \mathrm{wMB}(\mathbb{P}) \;:\Longleftrightarrow\; \exists Q \in \mathbb{P}\, ([Q] \subseteq A \;\vee\; [Q] \cap A = \varnothing)$$

- If $\mathbb{P}$ is **topological**, then we define

$$\mathrm{BP}(\mathbb{P}) := \{A \mid A \text{ has the Baire property in } (\omega^\omega, \mathbb{P})\}$$

Pointwise View of Determinacy

In “The pointwise view of determinacy”, the following results were proved: For **non-topological** $\mathbb{P}$:

1. $\mathbf{D} \not\to \mathsf{MB}(\mathbb{P})$ pointwise **(**i.e., $\mathbf{D} \not\subseteq \mathsf{MB}(\mathbb{P})$**)**.
2. $\mathbb{P}$ classified into three cases:

   Case 1: $\mathbf{D} \to \mathsf{wMB}(\mathbb{P})$ pointwise.

   Case 2: $\mathbf{D} \not\to \mathsf{wMB}(\mathbb{P})$ pointwise.

   Case 3: There are examples either way.

Pointwise View of Determinacy

In "The pointwise view of determinacy", the following results were proved: For **non-topological** $\mathbb{P}$:

1. **D** $\not\to$ MB$(\mathbb{P})$ pointwise (i.e., **D** $\not\subseteq$ MB$(\mathbb{P})$).
2. $\mathbb{P}$ classified into three cases:

   Case 1: **D** $\to$ wMB$(\mathbb{P})$ pointwise.

   Case 2: **D** $\not\to$ wMB$(\mathbb{P})$ pointwise.

   Case 3: There are examples either way.

Question: Can the same analyzis be done for **topological** $\mathbb{P}$ and BP$(\mathbb{P})$? What about wBP$(\mathbb{P})$?

Baire Property

Theorem: If $\mathbb{P}$ is **non-atomic** then $\mathbf{D} \not\to \mathsf{BP}(\mathbb{P})$ pointwise.

Baire Property

Theorem: If $\mathbb{P}$ is **non-atomic** then **D** $\not\to$ BP$(\mathbb{P})$ pointwise.

Proof:

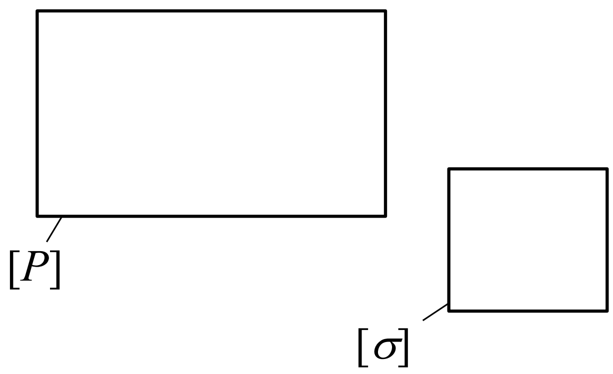
- If $A \in$ BP$(\mathbb{P})$ then for every open O there is a perfect tree T in O such that $[T] \subseteq A$ or $[T] \cap A = \varnothing$.

Baire Property

Theorem: If $\mathbb{P}$ is **non-atomic** then **D** $\not\to$ BP$(\mathbb{P})$ pointwise.

Proof:

- If $A \in$ BP$(\mathbb{P})$ then for every open O there is a perfect tree T in O such that $[T] \subseteq A$ or $[T] \cap A = \varnothing$.
- Find a $P \in \mathbb{P}$ and a strategy σ such that $[P] \cap [\sigma] = \varnothing$.

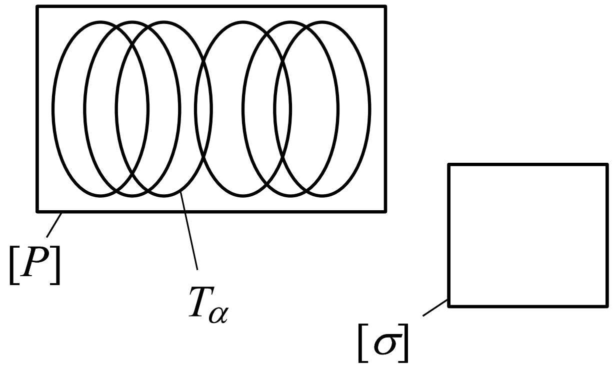


Baire Property

Theorem: If $\mathbb{P}$ is **non-atomic** then **D** $\not\to$ BP$(\mathbb{P})$ pointwise.

Proof:

- If $A \in$ BP$(\mathbb{P})$ then for every open O there is a perfect tree T in O such that $[T] \subseteq A$ or $[T] \cap A = \varnothing$.
- Find a $P \in \mathbb{P}$ and a strategy σ such that $[P] \cap [\sigma] = \varnothing$.
- Let $\langle T_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ enumerate all perfect trees in $[P]$.



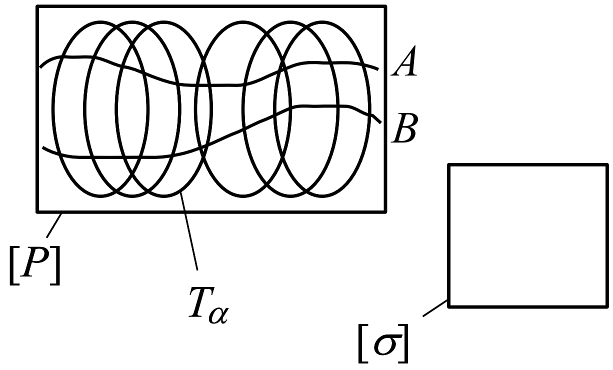
Baire Property

Theorem: If $\mathbb{P}$ is **non-atomic** then **D** $\not\to$ BP$(\mathbb{P})$ pointwise.

Proof:

- If $A \in$ BP$(\mathbb{P})$ then for every open O there is a perfect tree T in O such that $[T] \subseteq A$ or $[T] \cap A = \varnothing$.
- Find a $P \in \mathbb{P}$ and a strategy σ such that $[P] \cap [\sigma] = \varnothing$.
- Let $\langle T_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ enumerate all perfect trees in $[P]$.
- Since also $|T_\alpha| = 2^{\aleph_0}$, we find two Bernstein components A and B with $A \cap B = \varnothing$ and

$$\forall \alpha < 2^{\aleph_0} \ (A \cap [T_\alpha] \neq \varnothing \ \wedge \ B \cap [T_\alpha] \neq \varnothing)$$

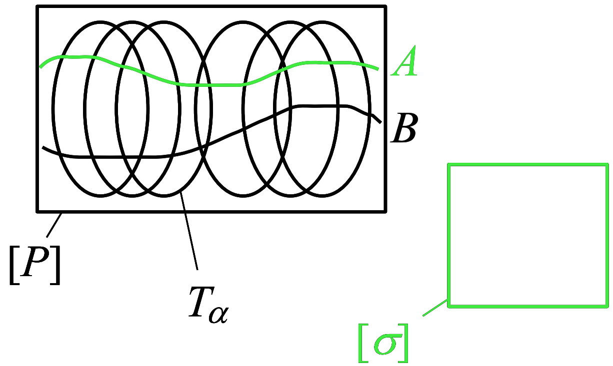


Baire Property

Theorem: If $\mathbb{P}$ is **non-atomic** then **D** $\not\to$ BP$(\mathbb{P})$ pointwise.

Proof:

- If $A \in$ BP$(\mathbb{P})$ then for every open O there is a perfect tree T in O such that $[T] \subseteq A$ or $[T] \cap A = \varnothing$.
- Find a $P \in \mathbb{P}$ and a strategy σ such that $[P] \cap [\sigma] = \varnothing$.
- Let $\langle T_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ enumerate all perfect trees in $[P]$.
- Since also $|T_\alpha| = 2^{\aleph_0}$, we find two Bernstein components A and B with $A \cap B = \varnothing$ and
$$\forall \alpha < 2^{\aleph_0} \ (A \cap [T_\alpha] \neq \varnothing \ \wedge \ B \cap [T_\alpha] \neq \varnothing)$$
- Let $A' := A \cup [\sigma]$. Then for **no** perfect tree T in $[P]$ do we have $[T] \subseteq A'$ or $[T] \cap A' = \varnothing$, so neither A' nor its complement is in BP$(\mathbb{P})$. But either A' or its complement is determined. $\square$



Weak Baire property

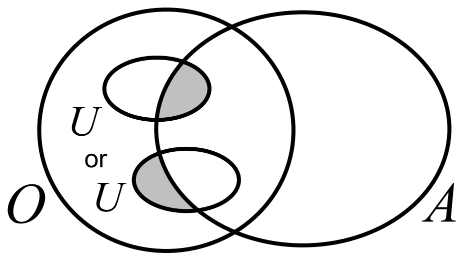
How to define wBP$(\mathbb{P})$?

Weak Baire property

How to define wBP$(\mathbb{P})$?

Use the following fact: A has the Baire property iff

$$\forall O\ \exists U \subseteq O\ (U \cap A \text{ is meager} \ \vee\ U \setminus A \text{ is meager})$$

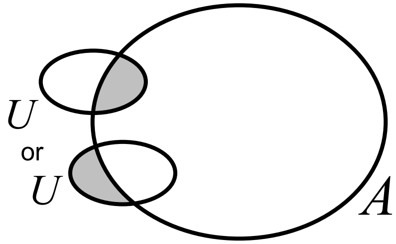


Weak Baire property

How to define wBP$(\mathbb{P})$?

Use the following fact: A has the Baire property iff

$$\forall O\ \exists U \subseteq O\ (U \cap A \text{ is meager } \vee\ U \setminus A \text{ is meager})$$



Definition: A has the weak Baire property iff

$$\exists U\ (U \cap A \text{ is meager } \vee\ U \setminus A \text{ is meager})$$

Three Cases

Consider the topological space $(\omega^\omega, \mathbb{P})$ or $(2^\omega, \mathbb{P})$.

Case 1: For every σ:

$$\exists P \in \mathbb{P} \text{ s.t. } [P] \setminus [\sigma] \text{ is meager}$$

Case 2: For some σ:

$$\forall P \in \mathbb{P}\ \exists Q \leq P \text{ s.t. } [Q] \cap [\sigma] \text{ is meager}$$

Case 3: None of the above.

Three Cases

Consider the topological space $(\omega^\omega, \mathbb{P})$ or $(2^\omega, \mathbb{P})$.

Case 1: For every σ:

$$\exists P \in \mathbb{P} \text{ s.t. } [P] \setminus [\sigma] \text{ is meager}$$

Case 2: For some σ:

$$\forall P \in \mathbb{P}\, \exists Q \leq P \text{ s.t. } [Q] \cap [\sigma] \text{ is meager}$$

Case 3: None of the above.

Case 1: D $\rightarrow$ wBP$(\mathbb{P})$ pointwise.

Case 2: D $\not\rightarrow$ wBP$(\mathbb{P})$ pointwise.

Case 3: ?

Three Cases

Consider the topological space $(\omega^\omega, \mathbb{P})$ or $(2^\omega, \mathbb{P})$.

Case 1: For every σ:

$$\exists P \in \mathbb{P} \text{ s.t. } [P] \setminus [\sigma] \text{ is meager}$$

Case 2: For some σ:

$$\forall P \in \mathbb{P}\ \exists Q \leq P \text{ s.t. } [Q] \cap [\sigma] \text{ is meager}$$

Case 3: None of the above.

Case 1: D $\rightarrow$ wBP$(\mathbb{P})$ pointwise.

Case 2: D $\not\rightarrow$ wBP$(\mathbb{P})$ pointwise.

All standard $\mathbb{P}$ *belong to this category*

Case 3: ?

Asymmetric Properties

Perfect Set Property: A is countable or contains a perfect tree. This is an asymmetric property, saying that "A is big or small".

PSP $=$ "Asym$(\mathbb{S})$" (Sacks forcing).

Analogous asymmetric properties have been defined for other $(\mathbb{P}, \leq)$.

Asymmetric Properties

Perfect Set Property: A is countable or contains a perfect tree. This is an asymmetric property, saying that "A is big or small".

PSP $=$ "Asym$(\mathbb{S})$" (Sacks forcing).

Analogous asymmetric properties have been defined for other $(\mathbb{P}, \leq)$.

Question:

1. Is there a general definition for Asym$(\mathbb{P})$?
2. What about **D** $\rightarrow$ Asym$(\mathbb{P})$ pointwise?

Notions of Smallness

Definition:

- For two reals x, y, define

$$x \leq^* y \;:\Longleftrightarrow\; \forall^\infty n\; (x(n) \leq y(n))$$

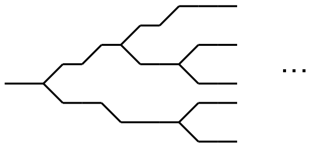
- $A \subseteq \omega^\omega$ is σ-**bounded** iff $\exists f\; \forall x \in A\; (x \leq^* f)$.
- $A \subseteq \omega^\omega$ is **dominating** iff $\forall f\; \exists x \in A\; (f \leq^* x)$.
- $A \subseteq \omega^\omega$ is **strongly dominating** iff

$$\forall f\; \exists x \in A\; \forall^\infty n\; [x(n+1) > f(x(n))]$$

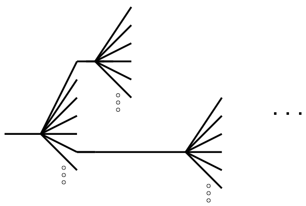
Notions of Largeness

Definition:

- **Perfect tree**: every node has an extension which is a splitting node.



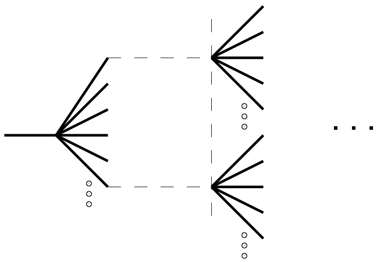
- **Super-perfect tree**: every splitting node is ω-splitting and every node has an extension which is an ω-splitting node.



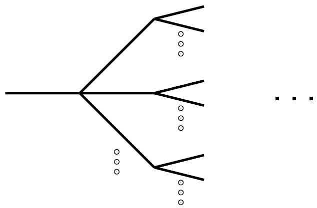
- **Spinas tree**: super-perfect tree such that for every $t \in T$:

$$\forall s_1, s_2 \ (t^\frown s_1 \text{ and } t^\frown s_2 \text{ are } \omega\text{-splitting nodes of } T \rightarrow |s_1| = |s_2|)$$

i.e., the next splitting node is a fixed distance away from t.



- **Laver tree**: the stem is finite, and after the stem, every node is ω-splitting.



Asymmetric properties

Forcing	Largness	Smallness	**Reg**
$\mathbb{C}$	contains perfect tree	countable	PSP
$\mathbb{M}$	contains super-perfect tree	σ-bounded	K_σ-regularity
$\mathbb{L}^*$	contains Spinas tree	not dominating	u-regularity
$\mathbb{L}$	contains Laver tree	not strongly dominating	Laver-regularity

What do all these properties have in common?

Asymmetric games

Definition:

1. **Asymmetric Game** $G^{\bullet}(A)$:

$$\begin{array}{l|llllll} I: & s_0 & & s_1 & & \dots & \\ \hline II: & & n_1 & & n_2 & & \dots \end{array}$$

$s_i \in \omega^{<\omega} \setminus \{\varnothing\}, \quad n_i \in \omega$

I wins iff

- $\forall i \geq 1$: $s_i(0) \neq n_i$
- $x := s_0{}^\frown s_1{}^\frown s_2{}^\frown \cdots \in A$.

Asymmetric games

Definition:

1. **Asymmetric Game** $G^{\bullet}(A)$:

$$\begin{array}{c|cccccc} I: & s_0 & & s_1 & & \dots & \\ \hline II: & & n_1 & & n_2 & & \dots \end{array}$$

$s_i \in \omega^{<\omega} \setminus \{\varnothing\}, \quad n_i \in \omega$

I wins iff

- $\forall i \geq 1$: $s_i(0) \neq n_i$
- $x := s_0{}^\frown s_1{}^\frown s_2{}^\frown \cdots \in A$.

2. **Kechris Game** $\tilde{G}(A)$:

$$\begin{array}{c|cccccc} I: & s_0 & & s_1 & & \dots & \\ \hline II: & & n_1 & & n_2 & & \dots \end{array}$$

$s_i \in \omega^{<\omega} \setminus \{\varnothing\}, \quad n_i \in \omega$

I wins iff

- $\forall i \geq 1$: $s_i(0) \geq n_i$
- $x := s_0{}^\frown s_1{}^\frown s_2{}^\frown \cdots \in A$.

Asymmetric games

Definition:

3. Spinas Game $G_u(A)$:

$$\begin{array}{c|ccc} I: & (s_0, k_0) & (s_1, k_1) & \dots \\ \hline II: & n_1 & n_2 & \end{array}$$

$s_i \in \omega^{<\omega} \setminus \{\varnothing\}, \quad k_i \in \omega \setminus \{0\}, \quad n_i \in \omega$

I wins iff

- $\forall i \geq 1 : |s_i| = k_{i-1}$
- $\forall i \geq 1$: $s_i(0) \geq n_i$
- $x := s_0{}^\frown s_1{}^\frown s_2{}^\frown \cdots \in A$.

Asymmetric games

Definition:

3. Spinas Game $G_u(A)$:

$$\begin{array}{c|ccccc} I: & (s_0,k_0) & & (s_1,k_1) & & \dots \\ \hline II: & & n_1 & & n_2 & \end{array}$$

$s_i \in \omega^{<\omega}\setminus\{\varnothing\}, \quad k_i \in \omega\setminus\{0\}, \quad n_i \in \omega$

I wins iff

- $\forall i \geq 1 : |s_i| = k_{i-1}$
- $\forall i \geq 1$: $s_i(0) \geq n_i$
- $x := s_0{}^\frown s_1{}^\frown s_2{}^\frown \cdots \in A$.

4. Goldstern Game $D(A)$:

$$\begin{array}{c|cccccc} I: & s_0 & & k_1 & & k_2 & \\ \hline II: & & n_1 & & n_2 & & \dots \end{array}$$

$s_i \in \omega^{<\omega} \setminus \{\varnothing\}, \quad k_i \in \omega, \quad n_i \in \omega$

I wins iff

- $\forall i \geq 1$: $k_i > n_i$
- $x := s_0{}^\frown \langle k_1, k_2, \dots\rangle \in A$

Game Characterizations of Asym

Theorem: (Davis 1964; Kechris 1977; Spinas 1993; Goldstern et al 1995)

1. (a) I *wins* $G^{\bullet}(A) \iff A$ *contains a perfect tree.*

   (b) II *wins* $G^{\bullet}(A) \iff A$ *is countable.*

2. (a) I *wins* $\tilde{G}(A) \iff A$ *contains a super-perfect tree.*

   (b) II *wins* $\tilde{G}(A) \iff A$ *is* σ*-bounded.*

3. (a) I *wins* $G_u(A) \iff A$ *contains a Spinas tree.*

   (b) II *wins* $G_u(A) \iff A$ *is not dominating.*

4. (a) I *wins* $D(A) \iff A$ *contains a Laver tree.*

   (b) II *wins* $D(A) \iff A$ *is not strongly dominating.*

Generalyzed Asymmetric Games

Definition: Start with set of parameters $\Phi := (R, r^0, \{\Theta_n\}_{n\in\omega}, f)$ where

- $R \subseteq \mathscr{P}(\omega^{<\omega})$ is a countable set of **requirements**.
- $r^0 \subseteq \omega^{<\omega}$ is the **initial requirement**.
- The Θ_i are countable sets of **additional information**.
- $f : \bigcup_n \Theta_n \longrightarrow \mathscr{P}(R)$.

Then the game $G_\Phi(A)$ is defined as follows:

$$\begin{array}{c|ccccccc} I: & (s_0,\theta_0) & & (s_1,\theta_1) & & (s_2,\theta_2) & & \dots \\ \hline II: & & r_1 & & r_2 & & \dots & \end{array}$$

where $s_i \in \omega^{<\omega} \setminus \{\varnothing\}, \quad \theta_i \in \Theta_i, \quad r_i \in R \cap f(\theta_{i-1})$

I wins $G_\Phi(A)$ iff

- $s_0 \in r^0$
- $\forall i \geq 1 : s_i \in r_i$
- $x := s_0{}^\frown s_1{}^\frown s_2{}^\frown \cdots \in A$.

Asymmetric Game Characterizations

Definition: Let $(\mathbb{P}, \leq)$ be a forcing notion, and let G_Φ be a generalized asymmetric game. We say that G_Φ **represents** $\mathbb{P}$ iff

$$\forall A \ [I \text{ has a winning strategy in } G_\Phi(A) \iff \exists P \in \mathbb{P} \ ([P] \subseteq A)]$$

Asymmetric Game Characterizations

Definition: Let $(\mathbb{P}, \leq)$ be a forcing notion, and let G_Φ be a generalized asymmetric game. We say that G_Φ **represents** $\mathbb{P}$ iff

$$\forall A \ [I \text{ has a winning strategy in } G_\Phi(A) \iff \exists P \in \mathbb{P} \ ([P] \subseteq A)]$$

Definition: Suppose G_Φ represents $\mathbb{P}$. Then we define $\text{Asym}_\Phi(\mathbb{P})$ by

$$A \in \text{Asym}_\Phi(\mathbb{P}) \ :\Longleftrightarrow \ \exists P \in \mathbb{P} \ ([P] \subseteq A) \quad \vee \quad II \text{ wins } G_\Phi(A)$$

Asymmetric Game Characterizations

Definition: Let $(\mathbb{P}, \leq)$ be a forcing notion, and let G_Φ be a generalized asymmetric game. We say that G_Φ **represents** $\mathbb{P}$ iff

$$\forall A \;\; [I \text{ has a winning strategy in } G_\Phi(A) \iff \exists P \in \mathbb{P} \; ([P] \subseteq A)]$$

Definition: Suppose G_Φ represents $\mathbb{P}$. Then we define $\text{Asym}_\Phi(\mathbb{P})$ by

$$A \in \text{Asym}_\Phi(\mathbb{P}) \;\; :\iff \;\; \exists P \in \mathbb{P} \; ([P] \subseteq A) \quad \vee \quad II \text{ wins } G_\Phi(A)$$

Examples:

- Let $\Phi_\bullet$ be the parameters corresponding to Davis's asymmetric game $G^\bullet(A)$. Then Sacks forcing is represented by $G_{\Phi_\bullet}(A)$. Therefore we can write PSP $=$ $\text{Asym}_{\Phi_\bullet}(\mathbb{S})$.
- Let $\Phi_\sim$ be the parameters corresponding to the Kechris game $\tilde{G}(A)$. Then Miller forcing is represented by $G_{\Phi_\sim}(A)$. Therefore K_σ-regularity $=$ $\text{Asym}_{\Phi_\sim}(\mathbb{M})$.

Questions

- Does every $\mathbb{P}$ have an asymmetric game G_Φ which represents it?
- If so, is the representation unique?

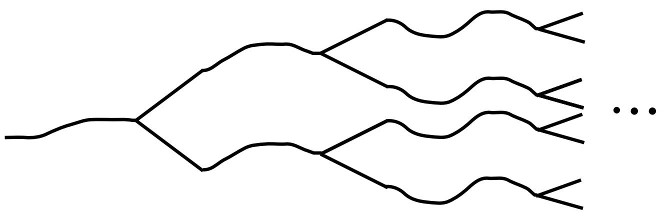
Questions

- Does every $\mathbb{P}$ have an asymmetric game G_Φ which represents it?
- If so, is the representation unique?

For example, Silver forcing probably doesn't have a game representation.

Silver forcing: **uniform trees**. Perfect trees T on 2^ω such that

$$\forall s, t \in T\ (|s| = |t| \rightarrow \{i \mid s^\frown\langle i\rangle \in T\} = \{i \mid t^\frown\langle i\rangle \in T\})$$



Pointwise Non-implication

Recall the question: "**D** $\longrightarrow$ Asym$(\mathbb{P})$ pointwise?"

Pointwise Non-implication

Recall the question: "**D** $\longrightarrow$ Asym$(\mathbb{P})$ pointwise?"

Theorem: Suppose $\mathbb{P}$ is represented by G_Φ which is non-trivial in the sense that I may choose between s and t with $s \perp t$. Then **D** $\not\rightarrow$ Asym$_\Phi(\mathbb{P})$ pointwise.

Pointwise Non-implication

Recall the question: "**D** $\rightarrow$ Asym$(\mathbb{P})$ pointwise?"

Theorem: Suppose $\mathbb{P}$ is represented by G_Φ which is non-trivial in the sense that I may choose between s and t with $s \perp t$. Then **D** $\not\rightarrow$ Asym$_\Phi(\mathbb{P})$ pointwise.

Proof:

Pointwise Non-implication

Recall the question: "**D** $\to$ Asym$(\mathbb{P})$ pointwise?"

Theorem: Suppose $\mathbb{P}$ is represented by G_Φ which is non-trivial in the sense that I may choose between s and t with $s \perp t$. Then **D** $\not\to$ Asym$_\Phi(\mathbb{P})$ pointwise.

Proof:

- Fix $s \in \omega^{<\omega}$ which "I may play in the first move". Fix a σ with $s \notin \sigma$.

Pointwise Non-implication

Recall the question: "**D** $\rightarrow$ Asym$(\mathbb{P})$ pointwise?"

Theorem: Suppose $\mathbb{P}$ is represented by G_Φ which is non-trivial in the sense that I may choose between s and t with $s \perp t$. Then **D** $\not\rightarrow$ Asym$_\Phi(\mathbb{P})$ pointwise.

Proof:

- Fix $s \in \omega^{<\omega}$ which "I may play in the first move". Fix a σ with $s \notin \sigma$.
- Let $\langle T_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ enumerate perfect trees and $\langle \tau_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ II-strategies. Let $K_\alpha := \{x \mid s \subseteq x \ \wedge \ x$ is a play according to $\tau_\alpha\}$

Pointwise Non-implication

Recall the question: "**D** $\to$ Asym$(\mathbb{P})$ pointwise?"

Theorem: Suppose $\mathbb{P}$ is represented by G_Φ which is non-trivial in the sense that I may choose between s and t with $s \perp t$. Then **D** $\not\to$ Asym$_\Phi(\mathbb{P})$ pointwise.

Proof:

- Fix $s \in \omega^{<\omega}$ which "I may play in the first move". Fix a σ with $s \notin \sigma$.
- Let $\langle T_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ enumerate perfect trees and $\langle \tau_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ II-strategies. Let $K_\alpha := \{x \mid s \subseteq x \ \wedge \ x$ is a play according to $\tau_\alpha\}$
- By the non-triviality assumption, $|K_\alpha| = 2^{\aleph_0}$ for all α.

Pointwise Non-implication

Recall the question: "**D** $\to$ Asym$(\mathbb{P})$ pointwise?"

Theorem: Suppose $\mathbb{P}$ is represented by G_Φ which is non-trivial in the sense that I may choose between s and t with $s \perp t$. Then **D** $\not\to$ Asym$_\Phi(\mathbb{P})$ pointwise.

Proof:

- Fix $s \in \omega^{<\omega}$ which "I may play in the first move". Fix a σ with $s \notin \sigma$.
- Let $\langle T_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ enumerate perfect trees and $\langle \tau_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ II-strategies. Let $K_\alpha := \{x \mid s \subseteq x \ \wedge \ x$ is a play according to $\tau_\alpha\}$
- By the non-triviality assumption, $|K_\alpha| = 2^{\aleph_0}$ for all α.
- Inductively find Bernstein components A and B as follows: given A_α, B_α, choose
  - $a_{\alpha+1} \in K_\alpha \setminus (A_\alpha \cup B_\alpha)$
  - $b_{\alpha+1} \in [T_\alpha] \setminus (A_\alpha \cup B_\alpha \cup \{a_{\alpha+1}\})$

  and let $A_{\alpha+1} := A_\alpha \cup \{a_{\alpha+1}\}$ and $B_{\alpha+1} := B_\alpha \cup \{b_{\alpha+1}\}$.

Pointwise Non-implication

Recall the question: "**D** $\to$ Asym$(\mathbb{P})$ pointwise?"

Theorem: Suppose $\mathbb{P}$ is represented by G_Φ which is non-trivial in the sense that I may choose between s and t with $s \perp t$. Then **D** $\not\to$ Asym$_\Phi(\mathbb{P})$ pointwise.

Proof:

- Fix $s \in \omega^{<\omega}$ which "I may play in the first move". Fix a σ with $s \notin \sigma$.
- Let $\langle T_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ enumerate perfect trees and $\langle \tau_\alpha \mid \alpha < 2^{\aleph_0} \rangle$ II-strategies. Let $K_\alpha := \{x \mid s \subseteq x \ \wedge \ x$ is a play according to $\tau_\alpha\}$
- By the non-triviality assumption, $|K_\alpha| = 2^{\aleph_0}$ for all α.
- Inductively find Bernstein components A and B as follows: given A_α, B_α, choose
    - $a_{\alpha+1} \in K_\alpha \setminus (A_\alpha \cup B_\alpha)$
    - $b_{\alpha+1} \in [T_\alpha] \setminus (A_\alpha \cup B_\alpha \cup \{a_{\alpha+1}\})$

  and let $A_{\alpha+1} := A_\alpha \cup \{a_{\alpha+1}\}$ and $B_{\alpha+1} := B_\alpha \cup \{b_{\alpha+1}\}$.
- Then $A \cap [\sigma] = \varnothing$ so A is determined; A doesn't contain a perfect tree by construction; II doesn't have a winning strategy in $G_\Phi(A)$ because $\forall \tau_\alpha$, we have $a_{\alpha+1} \in A$ according to τ_α. $\square$

Conclusion

There might be a better definition of Asym$(\mathbb{P})$, but as long as Asym$(\mathbb{P}) =$ Asym$_\Phi(\mathbb{P})$ for some non-trivial G_Φ representing $\mathbb{P}$, we have

Conclusion: D $\not\to$ Asym$(\mathbb{P})$ pointwise.

In particular **D** $\not\to$ PSP, K_σ-regularity, u-regularity and Laver-regularity pointwise.

Thank you!

Yurii Khomskii
yurii@deds.nl