Polarized Partition Properties on the Second Level of the Projective Hierarchy.

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(Lebesgue measurability, Baire property, Ramsey property, Marczewski measurability)

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"*More regularity on $\mathbf{\Delta}^1_2/\mathbf{\Sigma}^1_2$-level* $\quad \propto \quad$ L *gets smaller*"

Examples

1. $\mathbf{\Delta}^1_2(\text{Lebesgue}) \iff \forall a \; \exists$ random-generic$/\mathrm{L}[a]$
2. $\mathbf{\Delta}^1_2(\text{Baire Property}) \iff \forall a \; \exists$ Cohen-generic$/\mathrm{L}[a]$

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2. $\mathbf{\Delta}^1_2$(Baire Property) $\iff \forall a\ \exists$ Cohen-generic$/\mathrm{L}[a]$
3. $\mathbf{\Delta}^1_2$(Ramsey) $\iff \forall a\ \exists$ Ramsey real $/\mathrm{L}[a]$
4. $\mathbf{\Delta}^1_2$(Laver) $\iff \forall a\ \exists$ dominating real $/\mathrm{L}[a]$
5. $\mathbf{\Delta}^1_2$(Miller) $\iff \forall a\ \exists$ unbounded real $/\mathrm{L}[a]$
6. $\mathbf{\Delta}^1_2$(Sacks) $\iff \forall a\ \exists$ real $\notin \mathrm{L}[a]$

Where

- $x \in [\omega]^\omega$ is *Ramsey* over $\mathrm{L}[a]$ if for all $A \subseteq [\omega]^2 \cap \mathrm{L}[a]\ \exists n$ s.t. $[x \setminus n]^2 \subseteq A$ or $[x \setminus n]^2 \subseteq ([\omega]^2 \setminus A)$
- $x \in \omega^\omega$ is *dominating* over $\mathrm{L}[a]$ if $\forall y \in \omega^\omega \cap \mathrm{L}[a]\ \forall^\infty n(y(n) < x(n))$
- $x \in \omega^\omega$ is *unbounded* over $\mathrm{L}[a]$ if $\forall y \in \omega^\omega \cap \mathrm{L}[a]\ \exists^\infty n(y(n) < x(n))$

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(Non-)implications

Given two regularity properties: Reg_1 and Reg_2, we are interested in:

$$\mathbf{\Gamma}_1(\text{Reg}_1) \implies \mathbf{\Gamma}_2(\text{Reg}_2)?$$

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- Positive answer: find a ZFC-proof
- Negative answer: find a model M s.t. $M \models \mathbf{\Gamma}_1(\mathrm{Reg}_1)$ but $M \models \neg\mathbf{\Gamma}_2(\mathrm{Reg}_2)$

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What has been established so far?

Diagram of implications

$$\begin{tikzcd}
 & \mathbf{\Sigma}^1_2(\mathbb{E}) = \mathbf{\Sigma}^1_2(\mathbb{D}) \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & & \mathbf{\Sigma}^1_2(\mathbb{B}) = \mathbf{\Delta}^1_2(\mathbb{A}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
\mathbf{\Sigma}^1_2(\mathbb{R}) = \mathbf{\Delta}^1_2(\mathbb{R}) \arrow[d, Rightarrow] & \mathbf{\Sigma}^1_2(\mathbb{C}) = \mathbf{\Delta}^1_2(\mathbb{D}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] & \mathbf{\Delta}^1_2(\mathbb{E}) \arrow[dl, Rightarrow] \arrow[dd, Rightarrow] & \mathbf{\Delta}^1_2(\mathbb{B}) \arrow[ddl, Rightarrow, bend right] \arrow[dddll, Rightarrow, bend left] \\
\mathbf{\Sigma}^1_2(\mathbb{L}) = \mathbf{\Delta}^1_2(\mathbb{L}) \arrow[ddd, Rightarrow] \arrow[drr, Rightarrow] & \mathbf{\Delta}^1_2(\mathbb{C}) \arrow[d, Rightarrow] & & \\
 & \mathbf{\Sigma}^1_2(\mathbb{V}) \arrow[ddl, Rightarrow] \arrow[dd, Rightarrow] & \text{ev. diff.} \arrow[dddll, Rightarrow, bend left] & \\
 & & & \\
\mathbf{\Sigma}^1_2(\mathbb{M}) = \mathbf{\Delta}^1_2(\mathbb{M}) \arrow[d, Rightarrow] & \mathbf{\Delta}^1_2(\mathbb{V}) \arrow[dl, Rightarrow] & & \\
\mathbf{\Sigma}^1_2(\mathbb{S}) = \mathbf{\Delta}^1_2(\mathbb{S}) & & &
\end{tikzcd}$$

Diagram: Brendle & Löwe, *Eventually different functions and inaccessible cardinals*

Some general theorems

Theorem (Ikegami, 2008) Let $\mathbb{P}$ be a proper, tree-like forcing on ω^ω, and $I_\mathbb{P}$ a canonical σ-ideal such that $\mathbb{P} \hookrightarrow_d$ BOREL$(\omega^\omega)/I_\mathbb{P}$. Moreover suppose that the membership of Borel sets in $I_\mathbb{P}$ is a $\mathbf{\Sigma}^1_2$ property. Call a set A $\mathbb{P}$*-measurable* if

$$\forall p \, \exists q \leq p \, ([q] \subseteq^* A \vee [q] \subseteq^* \omega^\omega \setminus A)$$

Then T.F.A.E.

1. $\mathbf{\Delta}^1_2(\mathbb{P}\text{-measurability})$
2. $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness
3. $\forall a \, \exists x$ quasi-$I_\mathbb{P}$-generic over $\mathrm{L}[a]$

where x is *quasi-$I_\mathbb{P}$-generic* over M if $x \notin B$ for all Borel sets $B \in I_\mathbb{P}$, coded in M.

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Theorem (Ikegami, 2008) With additional (technical) assumptions on the ideal $I_\mathbb{P}$, T.F.A.E.

1. $\mathbf{\Sigma}^1_2(\mathbb{P}$-measurability$)$
2. $\forall a\, \exists$co-$I_\mathbb{P}$ set of quasi-$I_\mathbb{P}$-generics over $\mathrm{L}[a]$

Polarized Partitions

Definition. Letters H, J etc. will denote infinite sequences of finite subsets of ω, i.e. $H : \omega \longrightarrow [\omega]^{<\omega}$. Use abbreviation: $[H] = \prod_{i\in\omega} H(i)$.

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- A set/partition $A \subseteq \omega^\omega$ satisfies the property $\begin{pmatrix} \omega \\ \omega \\ \dots \end{pmatrix} \to \begin{pmatrix} m_1 \\ m_2 \\ \dots \end{pmatrix}$

  (*unbounded polarized partition*) if

$$\exists H \text{ s.t. } \forall i\ |H(i)| = m_i \text{ and } [H] \subseteq A \text{ or } [H] \cap A = \varnothing$$

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  and $n_1, n_2, \dots$ are recursive in $m_1, m_2, \dots$.

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From now on, use generic notations $(\vec{\omega} \to \vec{m})$ and $(\vec{n} \to \vec{m})$.

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In [DiPrisco & Todorčević, 2003]:

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So, what about $\mathbf{\Delta}^1_2/\mathbf{\Sigma}^1_2(\vec{\omega} \to \vec{m})$ and $\mathbf{\Delta}^1_2/\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$?

Upper bound

Fact. $\Gamma(\text{Ramsey}) \implies \boldsymbol{\Gamma}(\vec{\omega} \to \vec{m})$.

Proof. Given A, let $X \in \omega^{\uparrow\omega}$ be homogeneous for $A \cap \omega^{\uparrow\omega}$. Then divide ran$(X)$ into $X_0, X_1, \ldots$ such that $|X_i| = m_i$. Now $H := \langle X_0, X_1, \ldots \rangle$ witnesses that A satisfies $(\vec{\omega} \to \vec{m})$.

Eventually different reals

Theorem. (Brendle) If $\mathbf{\Delta}^1_2(\vec{\omega} \longrightarrow \vec{m})$ then $\forall a$ there is an eventually different real over $\mathrm{L}[a]$.

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- Let $A := \{x \mid$ first n at which $x(n) = y_x(n)$ is even$\}$. This is $\Delta^1_2(a)$ using the fact that $<_{\mathrm{L}[a]}$ is $\Delta^1_2(a)$.

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- Let $A := \{x \mid$ first n at which $x(n) = y_x(n)$ is even$\}$. This is $\Delta^1_2(a)$ using the fact that $<_{\mathrm{L}[a]}$ is $\Delta^1_2(a)$.
- Let H be homogeneous for A, w.l.o.g. $[H] \subseteq A$. But if $x \in [H]$ then let us change finitely many digits of x to produce a new real x', such that the first n at which $x'(n) = y_x(n)$ is odd but still $x' \in [H]$. It is easy to see that $y_x = y_{x'}$, hence $x' \notin A$: contradiction. $\square$

Diagram of implications

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\begin{tikzcd}[column sep=small]
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Mathias model

Theorem. (Brendle-Kh) Let $\mathrm{L}^{\mathbb{R}_{\omega_1}}$ be the *Mathias model*, i.e., the ω_1-iteration with countable support of Mathias forcing starting from L. Then $\mathrm{L}^{\mathbb{R}_{\omega_1}} \models \mathbf{\Delta}^1_2(\text{Ramsey})$ but $\neg\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$.

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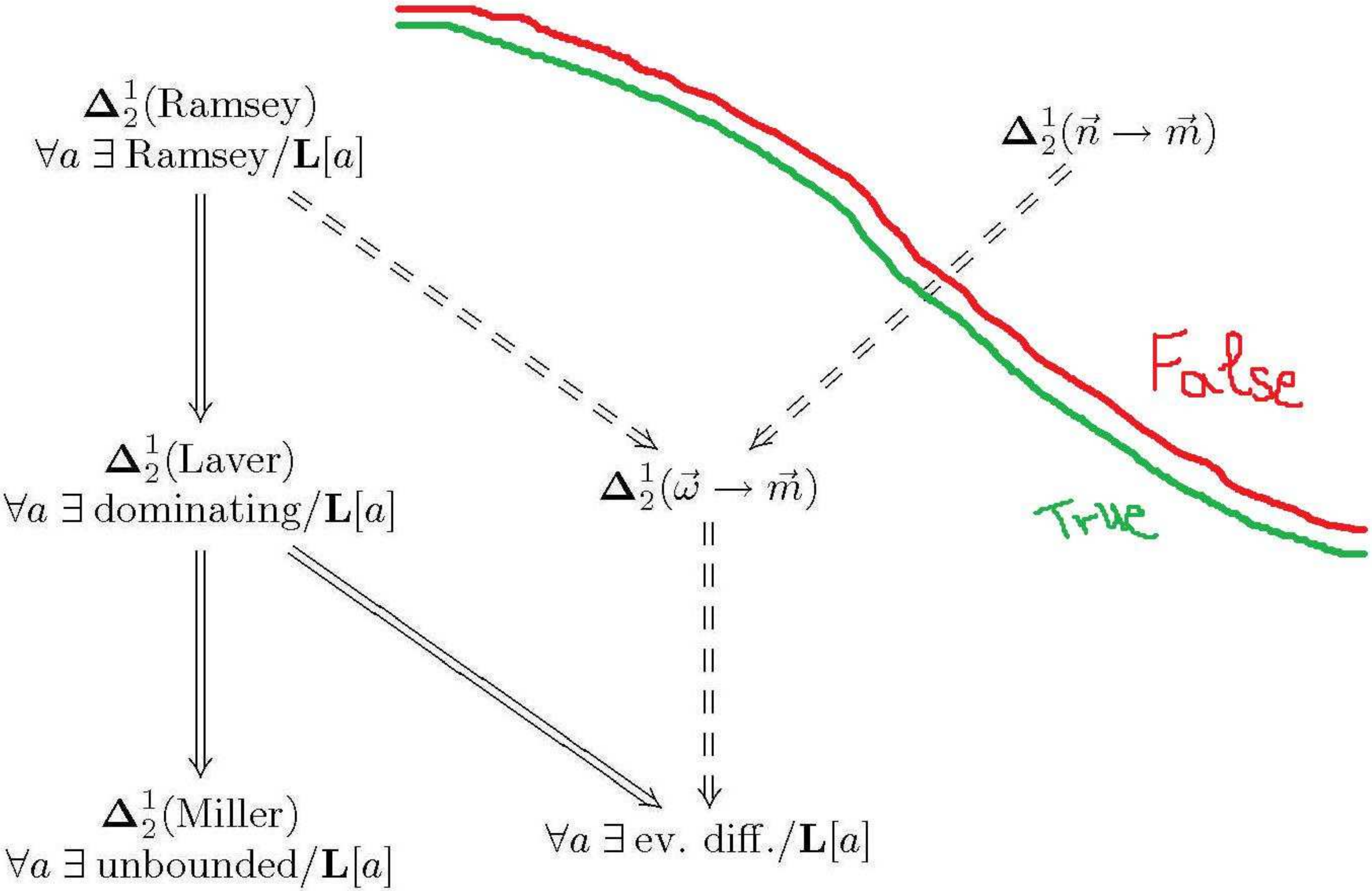

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- Use that to define a $\mathbf{\Delta}^1_2$ set A which explicitly violates $(\vec{n} \to \vec{m})$, where the m_i grow faster then 2^i. This set is well-defined because of the Laver property. $\square$

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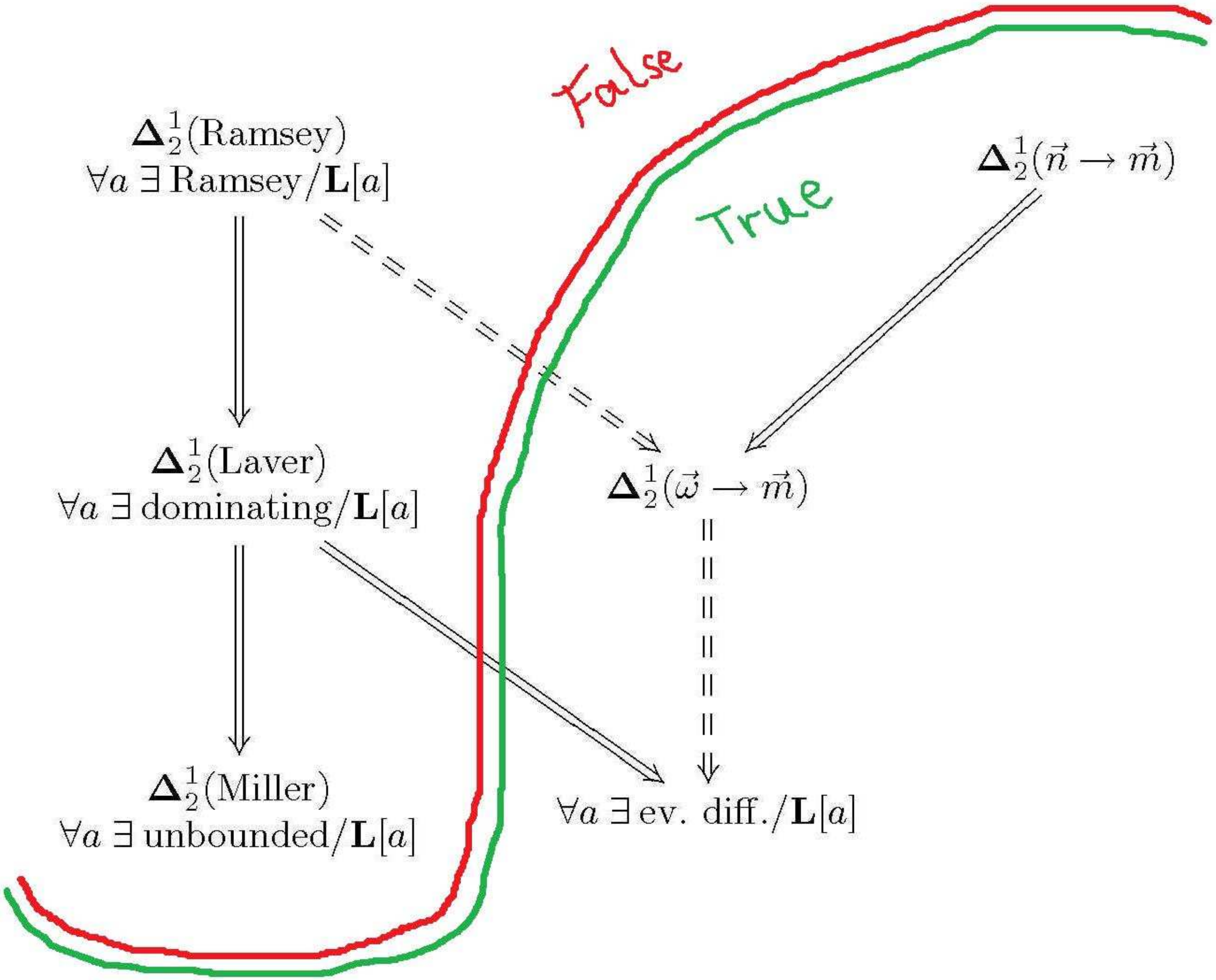

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2. If $\forall a$ there is a generic over $\mathrm{L}[a]$, then $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$ holds.

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  [To be precise: $F(n) \geq 2^{((2^{1/\epsilon_n})^n)}$]

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Remark: $\mathbb{P}_{\mathrm{KSZ}}$ adds a generic real, but the generic filter is not determined from the generic real in the usual way, and $\mathbb{P}_{\mathrm{KSZ}}$ is not in general representable as $\mathrm{BOREL}(\omega^\omega)/I$ for a σ-ideal I.

Proper and ω^ω-bounding

Theorem. (Kellner-Shelah, Shelah-Zapletal) If $\mathbb{P}_{\mathrm{KSZ}}$ is as above, and moreover

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The proof uses two properties from the general theory of creature forcings: for each n, $\mathbb{P}_n$ satisfies "ϵ_n*-bigness*" and "ϵ_n*-halving*".

Forcing $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$

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- Since q has empty stem, it witnesses that A satisfies $(\vec{n} \to \vec{m})$.

Forcing $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$

Corollary. An ω_1-iteration of $\mathbb{P}_{\mathrm{KSZ}}$, starting from L, gives a model in which $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$ holds but $\mathbf{\Delta}^1_2(\text{Miller})$ fails.

Notice that the bounds "$\vec{n}$" have been explicitly computed beforehand: they are the $F(n)$'s from the definition of $\mathbb{P}_{\mathrm{KSZ}}$.

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By another result of Zapletal, the conjunction "ω^ω-bounding and not adding splitting reals" is preserved in ω_1-iterations, so:

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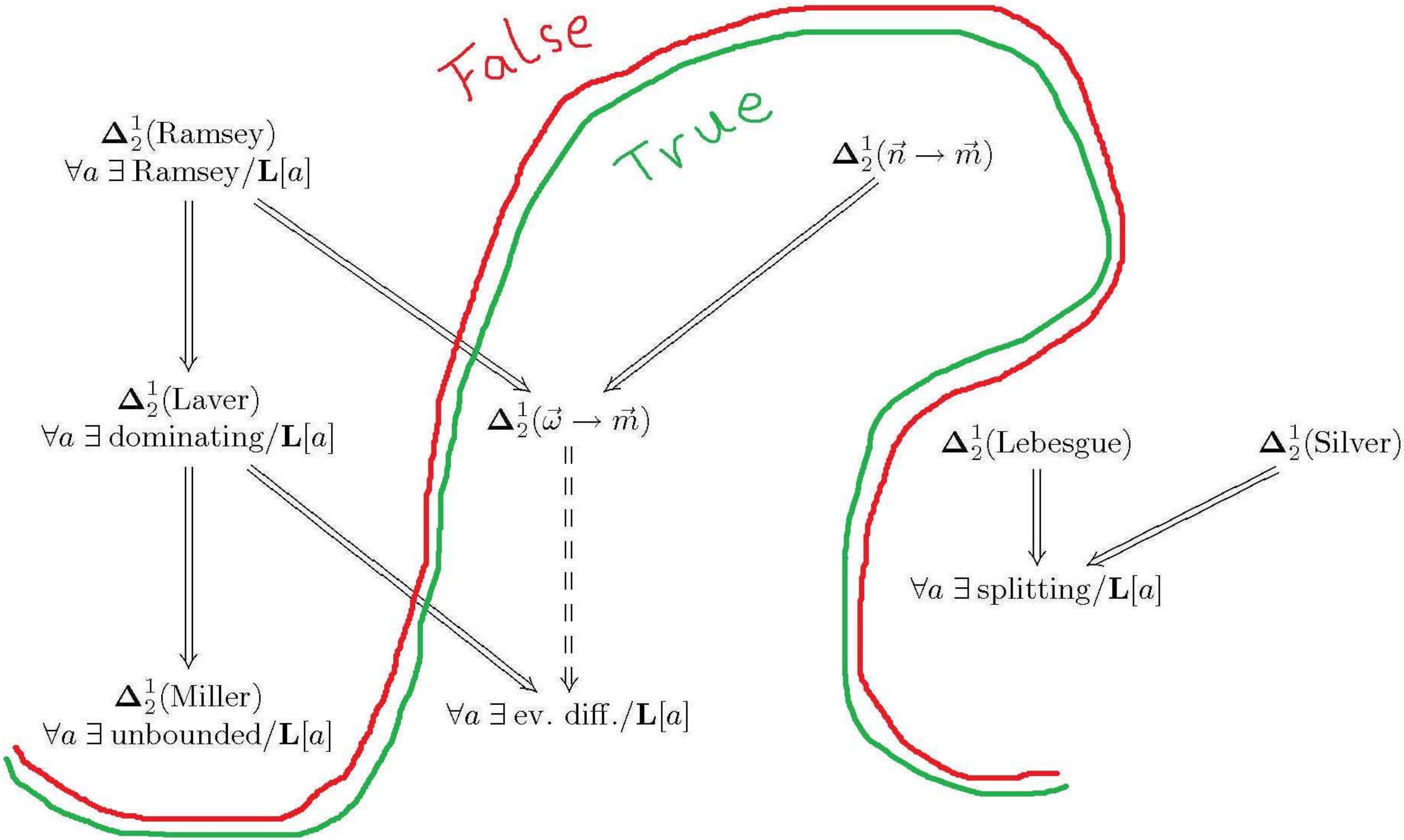

Open questions for $\mathbf{\Delta}^1_2$

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   Conjecture: $\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m})$ fails in the Random model.

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2. Is there a characterization of $\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m})$ and $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$ in terms of transcendence over L?

The property on the $\mathbf{\Sigma}^1_2$ level

Recall that for Ramsey, Sacks, Miller and Laver measurability, $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ are equivalent.

Question: Are $\mathbf{\Delta}^1_2$ and $\mathbf{\Sigma}^1_2$ equivalent for the polarized partition properties?

What we do know

Theorem. If $\mathbf{\Sigma}^1_2(\vec{\omega} \to \vec{m})$ then $\forall a \, \exists H$ s.t. $\forall x \in [H] : x$ is eventually different over $\mathrm{L}[a]$.

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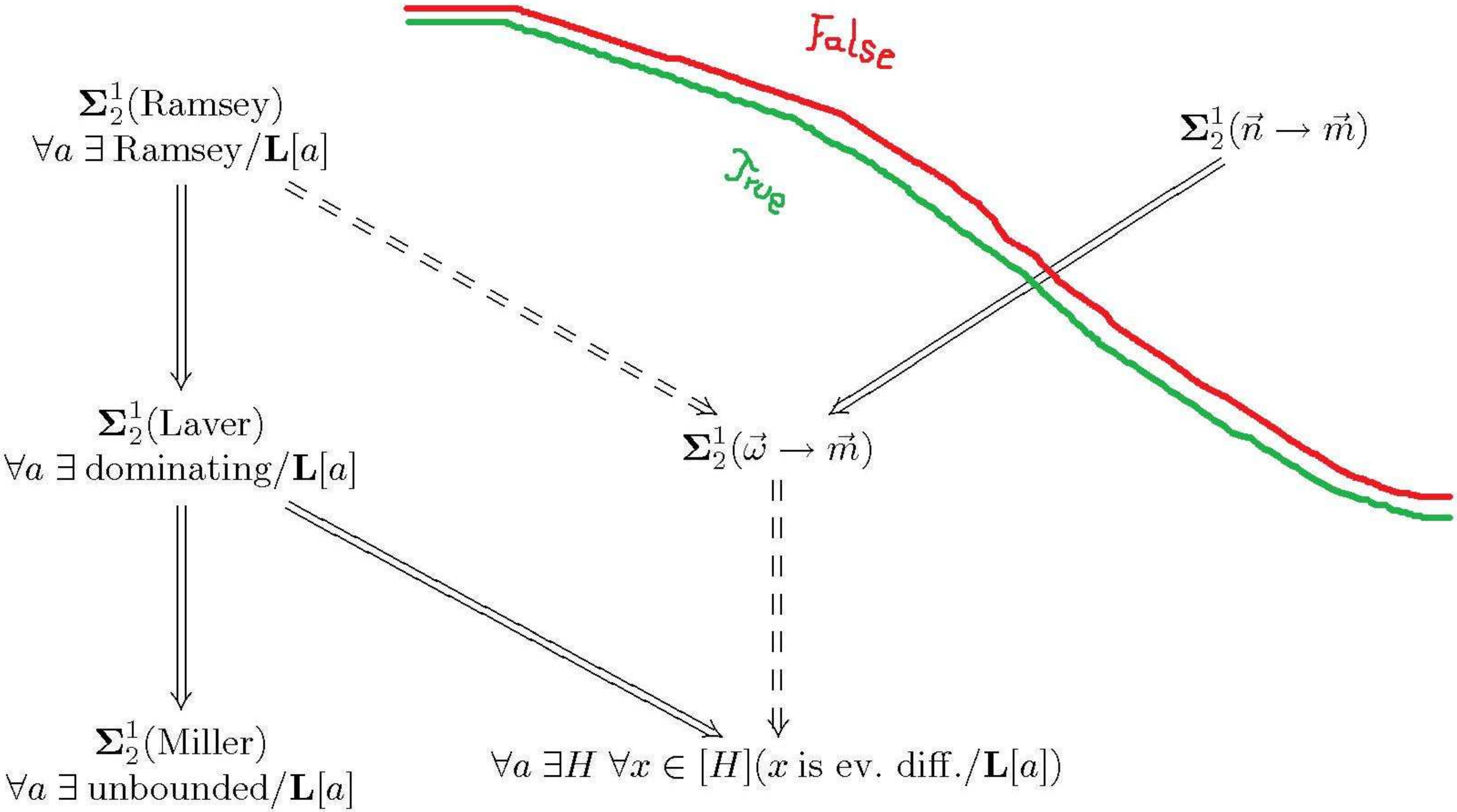
Theorem. In the Mathias model, $\mathbf{\Sigma}^1_2(\text{Ramsey})$ holds while $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ fails.

Diagram of implications



Forcing $\mathbf{\Sigma}^1_2(\vec{n} \longrightarrow \vec{m})$

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[DiPrisco & Todorčević] use a forcing $\mathbb{P}_{\mathrm{DPT}}$ adding a whole *generic product* H_G with the following property:

> For all Borel sets B in the ground model, $(*)$
> $B \cap [H_G]$ is relatively clopen in $[H_G]$.

Forcing $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$

Theorem. (Brendle-Kh) An ω_1-iteration of $\mathbb{P}_{\mathrm{DPT}}$ starting from L give a model where $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ holds.

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Proof.

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Corollary. There is a model where $\Sigma^1_2(\vec{n} \to \vec{m})$ holds but $\Sigma^1_2(\text{Miller})$ fails.

ありがとうございました