Regularity and Definability

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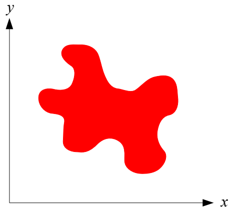

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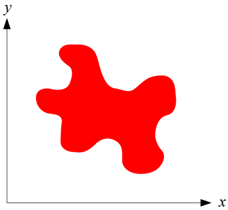

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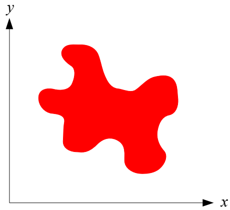

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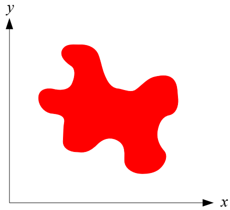

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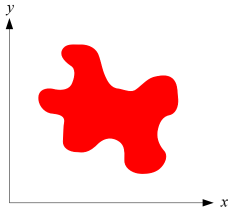
3. Relationship between these.
   - Independence from ZFC (forcing extensions over **L**).



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Captures the intuition of "size" or "volume" of a set of reals ("object in space").

Can naturally be extended to $\mathbb{R}^n$.

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Problematic consequences for spatial reasoning, e.g., Banach-Tarski paradox.

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Question

Can we find an explicit example of a non-regular set? (and what does that even mean?)

2. Definability

Descriptive set theory

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Complexity of $\mathbb{N}^2$-formulas: $\Sigma^0_n, \Pi^0_n, \ldots, \Sigma^1_n, \Pi^1_n, \ldots$.

Complexity of sets

Complexity of a set of reals measured by complexity of defining $\mathbb{N}^2$-formula.

$$A = \{x \in \mathbb{R} \mid \mathbb{N}^2 \models \phi(x, a)\}$$

Note that we allow a fixed real parameter $a \in \mathbb{R}$ in the definition.

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Relation with topology:

- $\boldsymbol{\Sigma}^0_1$ = open,
- $\boldsymbol{\Pi}^0_1$ = closed,
- $\boldsymbol{\Delta}^1_1$ = Borel,
- $\boldsymbol{\Sigma}^1_1$ = analytic (continuous image of Borel).

Hierarchy

$$
\begin{tikzcd}[column sep=small]
\overset{\text{(open)}}{\boldsymbol{\Sigma}^0_1} \arrow[dr, phantom, sloped, "\subseteq"] & & \boldsymbol{\Sigma}^0_2 \arrow[dr, phantom, sloped, "\subseteq"] & & & \overset{\text{(analytic)}}{\boldsymbol{\Sigma}^1_1} \arrow[dr, phantom, sloped, "\subseteq"] & & \boldsymbol{\Sigma}^1_2 \arrow[dr, phantom, sloped, "\subseteq"] & \\
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 & & \boldsymbol{\Delta}^0_2 & & & & \dots & & \boldsymbol{\Delta}^1_1 & & & & \boldsymbol{\Delta}^1_2 & & & & \dots \\
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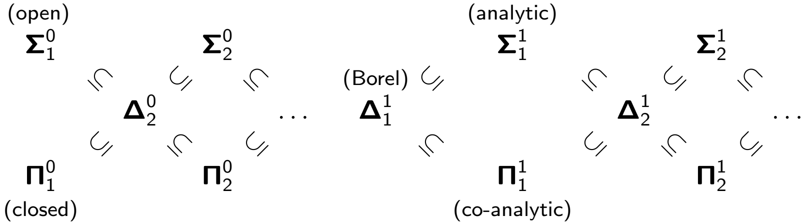

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So "paradoxes" cannot occur if we restrict attention to analytic/co-analytic sets.

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Answer: It is independent of ZFC!

3. Independence results

Constuctible universe and extensions

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  - If we add "many" reals, yes.
  - If we add "not so many" reals, perhaps not.
- In fact, we can say exactly which reals must be added to obtain regularity on $\mathbf{\Sigma}^1_2/\mathbf{\Delta}^1_2$ level.

Solovay-Judah-Shelah characterizations

Theorem (Judah-Shelah 1989)

The following are equivalent:

1. *All* $\mathbf{\Delta}^1_2$ *sets are Lebesgue-measurable,*
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Statements "all $\boldsymbol{\Sigma}^1_2$ ($\boldsymbol{\Delta}^1_2$) sets are regular" correspond to "transcendence over $\mathbf{L}$".

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Example 2. Cohen forcing adds Cohen-generic reals but not random-generic reals. Therefore, if we iterate Cohen forcing (for $\aleph_1$ steps), we get a model where all $\mathbf{\Delta}^1_2$ sets have the Baire property but not all $\mathbf{\Delta}^1_2$ sets are Lebesgue measurable.

Strength of measurability

On the other hand, some properties are stronger than others:

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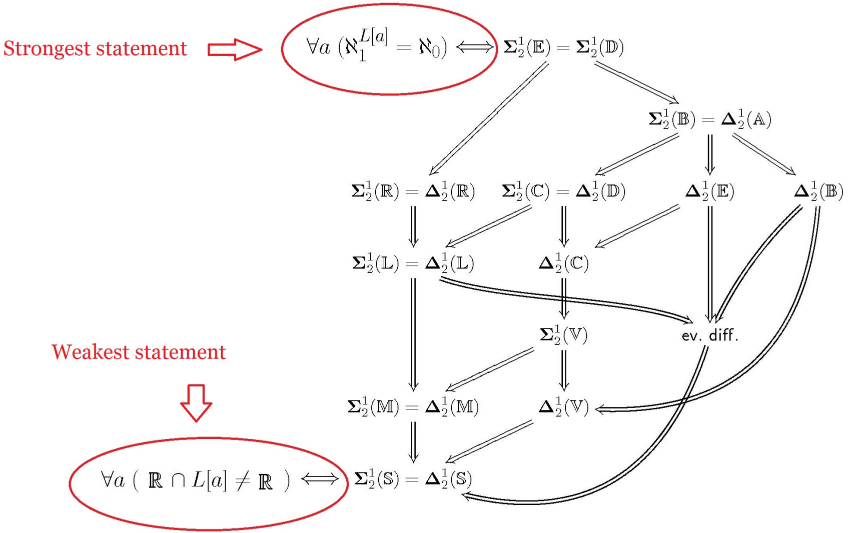

Brendle & Löwe, *Eventually different functions and inaccessible cardinals.*

$$
\begin{tikzcd}
\boldsymbol{\Sigma}^1_2(\text{Ramsey}) \arrow[dd, Rightarrow] \arrow[dr, Rightarrow, "(a)"] & & & \boldsymbol{\Sigma}^1_2(\vec{n} \to \vec{m}) \arrow[dll, Rightarrow, "(b)"'] \arrow[d, Rightarrow, "(c')"] & \\
& \boldsymbol{\Sigma}^1_2(\vec{\omega} \to \vec{m}) \arrow[d, Rightarrow, "(c)"'] \arrow[dddrr, Rightarrow, bend left, "(e')"] & & \boldsymbol{\Delta}^1_2(\vec{n} \to \vec{m}) \arrow[dll, Rightarrow, "(b')"'] \arrow[d, Rightarrow, "(d')"] & \\
\begin{array}{c} \boldsymbol{\Sigma}^1_2(\text{Laver}) \\ \forall a\ \exists\ \text{dom.}/\mathbf{L}[a] \end{array} \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \boldsymbol{\Delta}^1_2(\vec{\omega} \to \vec{m}) \arrow[d, Rightarrow, "(d)"'] \arrow[ddddrr, Rightarrow, bend left, "(e)"] & & \begin{array}{c} \forall a\ \exists\ \text{bounded} \\ \text{ev. diff.}/\mathbf{L}[a] \end{array} \arrow[dll, Rightarrow, "(b'')"'] & \\
\begin{array}{c} \boldsymbol{\Sigma}^1_2(\text{Miller}) \\ \forall a\ \exists\ \text{unb.}/\mathbf{L}[a] \end{array} & \forall a\ \exists\ \text{ev. diff.}/\mathbf{L}[a] & & & \\
& & & \begin{array}{c} \forall a\ \{x \mid x \text{ not } I_{E_0}\text{-} \\ \text{quasigeneric} \\ /\mathbf{L}[a]\} \in \mathbb{N}_{E_0} \end{array} \arrow[dd, Rightarrow] & \\
& & & & \boldsymbol{\Delta}^1_2(\text{doughnut}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \\
& & & \begin{array}{c} \forall a\ \exists\ I_{E_0}\text{-quasi-} \\ \text{generic}/\mathbf{L}[a] \end{array} & \forall a\ \exists\ \text{splitting}/\mathbf{L}[a]
\end{tikzcd}
$$

Brendle & Khomskii, *Polarized partitions on the second level of the projective hierarchy.*

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5. For some properties, whether it holds on the $\boldsymbol{\Sigma}^1_1$ or even Borel level is still open (e.g., does there exist a Borel maximal family of eventually different functions?)

Thank you!

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