Regularity and Definability

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The setting

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   - Baire property,
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2. **Definability.**
   - Classifying sets according to logical complexity.

3. **Relationship between these.**
   - Independence from ZFC (forcing extensions over \(L\)).
1. Regularity
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- For $q < q' \in \mathbb{Q}$, $\mu([q, q']) := q' - q$. 

Naturally extend to Borel subsets of $\mathbb{R}$. A $\subseteq \mathbb{R}$ is Lebesgue-null if $\exists B$ Borel with $A \subseteq B$ and $\mu(B) = 0$. A is Lebesgue-measurable if $\exists B$ Borel such that $(A \setminus B) \cup (B \setminus A)$ is Lebesgue-null. Captures the intuition of “size” or “volume” of a set of reals (“object in space”). Can naturally be extended to $\mathbb{R}^n$. 

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**Proof.**

If $A$ is Lebesgue-measurable then there exists a perfect set $P$ with $\mu(P) > 0$ s.t. $P \subseteq A$ or $P \cap A = \emptyset$. Use Axiom of Choice to diagonalize against perfect sets.
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Problematic consequences for spatial reasoning, e.g., Banach-Tarski paradox.
Other examples

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**Question**

Can we find an explicit example of a non-regular set? (and what does that even mean?)
2. Definability
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Focus on second-order number theory ($\mathbb{N}^2$):

- Variables range over natural numbers or real numbers.
- Natural number quantifiers: $\exists^0 \forall^0$,
- Real number quantifiers: $\exists^1 \forall^1$. 
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Complexity of $\mathbb{N}^2$-formulas: $\Sigma^0_n, \Pi^0_n, \ldots, \Sigma^1_n, \Pi^1_n, \ldots$. 
Complexity of sets

Complexity of a set of reals measured by complexity of defining $\mathbb{N}^2$-formula.

$$A = \{ x \in \mathbb{R} \mid \mathbb{N}^2 \models \phi(x, a) \}$$

Note that we allow a fixed real parameter $a \in \mathbb{R}$ in the definition.
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We say “$A$ has complexity $\Sigma_n^i (\Pi_n^i)$” iff $\phi$ has complexity $\Sigma_n^i (\Pi_n^i)$.
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We say “$A$ has complexity $\Sigma^i_n (\Pi^i_n)$” iff $\phi$ has complexity $\Sigma^i_n (\Pi^i_n)$.

Relation with topology:

- $\Sigma^0_1 = \text{open}$,
- $\Pi^0_1 = \text{closed}$,
- $\Delta^1_1 = \text{Borel}$,
- $\Sigma^1_1 = \text{analytic}$ (continuous image of Borel).
Hierarchy

(open) $\Sigma^0_1$ $\subseteq$ $\Sigma^0_2$ $\subseteq$ $\Sigma^1_0$ $\subseteq$ $\Sigma^1_1$ $\subseteq$ $\Sigma^1_2$ $\subseteq$ $\Delta^0_2$ $\subseteq$ $\Delta^0_1$ $\subseteq$ $\Delta^1_0$ $\subseteq$ $\Delta^1_1$ $\subseteq$ $\Delta^1_2$ $\subseteq$ $\Pi^0_1$ $\subseteq$ $\Pi^0_2$ $\subseteq$ $\Pi^1_0$ $\subseteq$ $\Pi^1_1$ $\subseteq$ $\Pi^1_2$ $\subseteq$ $\Sigma^0_1$ $\subseteq$ $\Sigma^0_2$ $\subseteq$ $\Sigma^1_0$ $\subseteq$ $\Sigma^1_1$ $\subseteq$ $\Sigma^1_2$ $\subseteq$ (Borel) $\Sigma^1_1$ $\subseteq$ $\Sigma^1_2$ $\subseteq$ (analytic) $\Sigma^1_1$ $\subseteq$ $\Sigma^1_2$ $\subseteq$ (closed) $\Pi^1_1$ $\subseteq$ $\Pi^1_2$ $\subseteq$ (co-analytic) $\Pi^1_1$ $\subseteq$ $\Pi^1_2$

All $\Sigma^1_1$ sets are regular.

For many properties, also all $\Pi^1_1$ sets are regular.

Irregular sets (produced by AC) may lie far outside this hierarchy.

So “paradoxes” cannot occur if we restrict attention to analytic/co-analytic sets.
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**Question:** Does the assertion “all $\Sigma^1_2$ sets are regular” hold?
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**Question:** Does the assertion “all $\Sigma_2^1$ sets are regular” hold?

**Answer:** It is independent of ZFC!
3. Independence results
Constructible universe and extensions

$L = \text{Gödel’s constructible universe.}$
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Constructible universe and extensions

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- There is a $\Sigma^1_2$-definable well-ordering of the continuum.

Forcing over $L$.

By forcing we can add new reals, destroy $\Sigma^1_2$ well-ordering. Does irregularity disappear?

If we add "many" reals, yes.

If we add "not so many" reals, perhaps not.

In fact, we can say exactly which reals must be added to obtain regularity on $\Sigma^1_2$ level.
Constructible universe and extensions

\[ L = \text{G"odel's constructible universe}. \]

- There is a \( \Sigma_2^1 \)-definable well-ordering of the continuum.
- Therefore irregularity exists on the \( \Sigma_2^1 \) (even \( \Delta_2^1 \)) level.

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Solovay-Judah-Shelah characterizations

Theorem (Judah-Shelah 1989)

The following are equivalent:

1. All $\Delta^1_2$ sets are Lebesgue-measurable,
2. For all $a \in \mathbb{R}$ there is a random-generic real over $L[a]$. 

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Iterated forcing extensions

Statements “all $\Sigma^1_2 (\Delta^1_2)$ sets are regular” correspond to “transcendence over $L$”.

Example 1. Random forcing adds random-generic reals but not Cohen-generic reals. Therefore, if we iterate random forcing for $\aleph_1$ steps, we get a model where all $\Delta^1_2$ sets are Lebesgue measurable, but not all $\Delta^1_2$ sets have the Baire property.

Example 2. Cohen forcing adds Cohen-generic reals but not random-generic reals. Therefore, if we iterate Cohen forcing (for $\aleph_1$ steps), we get a model where all $\Delta^1_2$ sets have the Baire property but not all $\Delta^1_2$ sets are Lebesgue measurable.
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Strength of measurability

On the other hand, some properties are stronger than others:

Theorem (Bartoszyński-Raisonnier-Stern 1984/1985)

If all $\Sigma^1_2$ sets are Lebesgue measurable then all $\Sigma^1_2$ sets have the Baire property.
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Strongest statement

\[ \forall a \left( V_{1}[a] = \mathbb{N}_0 \right) \iff \Sigma_2^1(\mathbb{E}) = \Sigma_2^1(\mathbb{D}) \]

\[ \Sigma_2^1(\mathbb{R}) = \Delta_2^1(\mathbb{R}) \]
\[ \Sigma_2^1(\mathbb{C}) = \Delta_2^1(\mathbb{D}) \]
\[ \Delta_2^1(\mathbb{E}) \]
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\[ \Sigma_2^1(\mathbb{V}) \]
\[ \Sigma_2^1(\mathbb{M}) = \Delta_2^1(\mathbb{M}) \]
\[ \Delta_2^1(\mathbb{V}) \]
\[ \forall a \left( \mathbb{R} \cap L[a] \neq \mathbb{R} \right) \iff \Sigma_2^1(\mathbb{S}) = \Delta_2^1(\mathbb{S}) \]

Weakest statement

\[ \Sigma_2^1(\mathbb{E}) = \Delta_2^1(\mathbb{A}) \]

Brendle & Löwe, *Eventually different functions and inaccessible cardinals.*
Brendle & Khomskii, *Polarized partitions on the second level of the projective hierarchy.*

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1. Given a regularity property, characterize it by transcendence property.

2. Given a transcendence property, characterize it by regularity.

3. Find general Solovay-Judah-Shelah-style theorems (some work done by Daisuke Ikegami; still many open questions).

4. Prove implications from $\Sigma^1_2/\Delta^1_2$ (Reg$_1$) to $\Sigma^1_2/\Delta^1_2$ (Reg$_2$), or produce a model which separates Reg$_1$ from Reg$_2$.

5. For some properties, whether it holds on the $\Sigma^1_1$ or even Borel level is still open (e.g., does there exist a Borel maximal family of eventually different functions?)
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Thank you!

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