Regularity Properties and Definability

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2. Definability: *what does logic have to do with it?*

3. Gödel, Solovay, Shelah: *. . . getting technical. . .*

1. Regularity Properties

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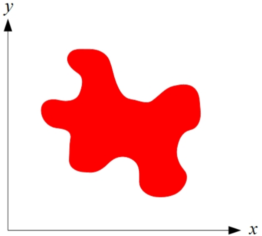

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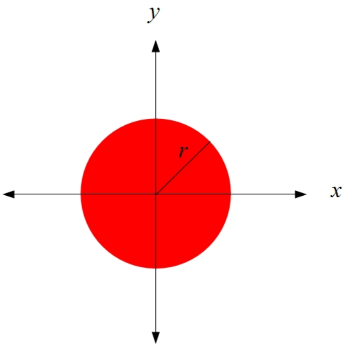

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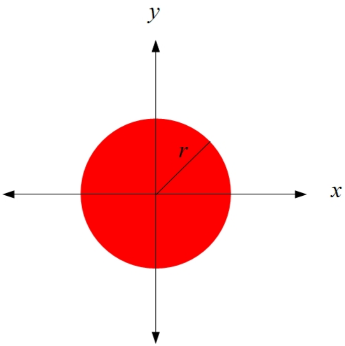
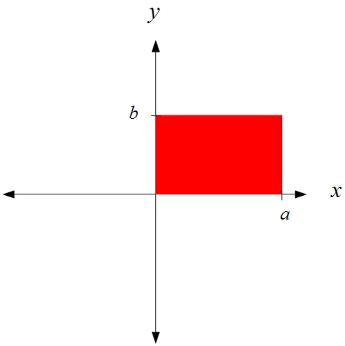


$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}$$

Examples



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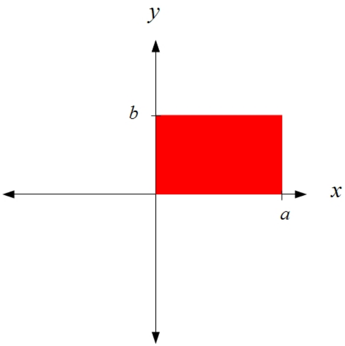

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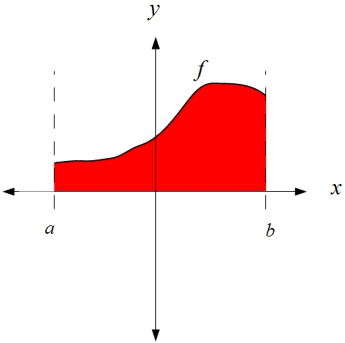

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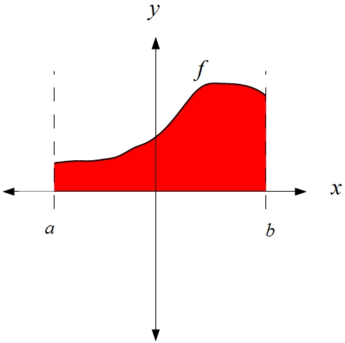

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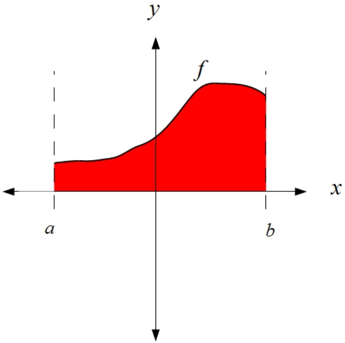

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Notice that from a set-theoretic point of view, all of the above “objects” are *subsets of the continuuum* (in this case, $A \subseteq \mathbb{R}^2$).

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- etc.

Regularity properties: isolated specifically to avoid such counter-intuitive constructions.

Lebesgue measure

Motivating example: Lebesgue measure.

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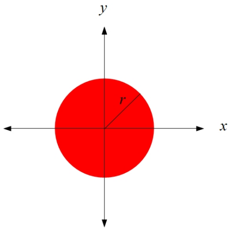

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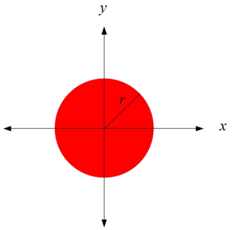

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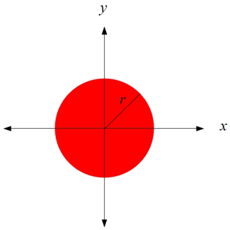

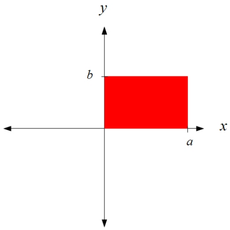

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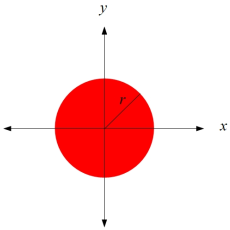

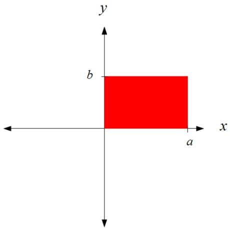

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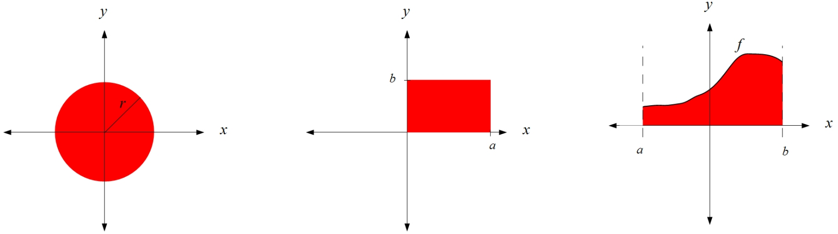

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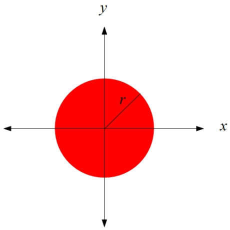

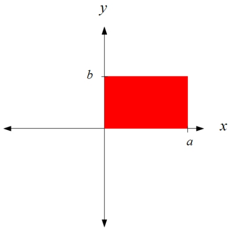

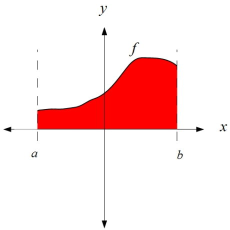

$\mu(A) = \pi \cdot r^2$

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$\mu(A) = \int_a^b f(x)dx$

Lebesgue measure (2)

Henri Lebesgue, in his PhD thesis from 1902, defined a precise way of measuring the "size" or "volume" of subsets $A \subseteq \mathbb{R}^n$ (needed for the definition of the *Lebesgue integral*).

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Easy modern proof.

Let U be an ultrafiler on ω. Identify $\mathcal{P}(\omega)$ with 2^ω, then U is non-Lebesgue-measurable as a subset of 2^ω (easy to translate to $\mathbb{R}$ or $\mathbb{R}^n$).

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Impossible? Mathematically possible, but the pieces are *not Lebesgue-measurable* (so our spatial intuition does not apply).

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2. Definability

Defining sets

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Complexity of $\mathbb{N}^2$-formulas: $\Sigma^0_n, \Pi^0_n, \dots, \Sigma^1_n, \Pi^1_n, \dots$ (number of alternating quantifiers).

Defining sets (2)

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Definition

If a set A can be written as $A = \{x \in \mathbb{R} \mid \mathbb{N}^2 \models \phi(x, a)\}$, then we say that A has complexity $\boldsymbol{\Sigma}^i_n$ ($\boldsymbol{\Pi}^i_n$) iff ϕ has complexity Σ^i_n (Π^i_n).

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Definition

A set A is *projective* if it is $\boldsymbol{\Sigma}^1_n$ or $\boldsymbol{\Pi}^1_n$ for some n.

Projective hierarchy

$$\begin{array}{cccccccccccccccc}
\boldsymbol{\Sigma}^0_1 & & & & \boldsymbol{\Sigma}^0_2 & & & & & \boldsymbol{\Sigma}^1_1 & & & & \boldsymbol{\Sigma}^1_2 & & \\
& \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \\
& & \boldsymbol{\Delta}^0_2 & & & & \cdots & \boldsymbol{\Delta}^1_1 & & & & \boldsymbol{\Delta}^1_2 & & & & \cdots \\
& \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
\boldsymbol{\Pi}^0_1 & & & & \boldsymbol{\Pi}^0_2 & & & & & \boldsymbol{\Pi}^1_1 & & & & \boldsymbol{\Pi}^1_2 & &
\end{array}$$

Projective hierarchy

$$
\begin{array}{cccccccccccccccc}
\text{(open)} & & & & & & & & & \text{(analytic)} & & & & & & \\
\boldsymbol{\Sigma}^0_1 & & & & \boldsymbol{\Sigma}^0_2 & & & & & \boldsymbol{\Sigma}^1_1 & & & & \boldsymbol{\Sigma}^1_2 & & \\
 & \subsetneq & & \subsetneq & & \subsetneq & & & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \\
 & & \boldsymbol{\Delta}^0_2 & & & & \cdots & \text{(Borel)}\ \boldsymbol{\Delta}^1_1 & & & & \boldsymbol{\Delta}^1_2 & & & & \cdots \\
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Connection with topology:

- $\boldsymbol{\Sigma}^0_1$ = open,
- $\boldsymbol{\Pi}^0_1$ = closed,
- $\boldsymbol{\Delta}^1_1$ = Borel,
- $\boldsymbol{\Sigma}^1_1$ = analytic (continuous image of Borel),
- $\boldsymbol{\Pi}^1_1$ = co-analytic (complement of analytic).

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All analytic sets are Lebesgue-measurable, satisfy the Property of Baire and the Perfect Set Property.

The same holds for **all other** (reasonable) regularity properties!
(proofs scattered throughout 20th century).

Back to the hierarchy . . .

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& \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \text{(Borel)} \;\rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & \\
& & \boldsymbol{\Delta}^0_2 & & & & \cdots & \boldsymbol{\Delta}^1_1 & & & \boldsymbol{\Delta}^1_2 & & & & \cdots \\
& \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & \\
\boldsymbol{\Pi}^0_1 & & & & \boldsymbol{\Pi}^0_2 & & & & \boldsymbol{\Pi}^1_1 & & & & \boldsymbol{\Pi}^1_2 & & \\
\text{(closed)} & & & & & & & & \text{(co-analytic)} & & & & & &
\end{array}
$$

OK!

No “weird things” or “paradoxes” can occur if we restrict attention to analytic sets.

Philosophical Intermezzo

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Why is there a connection between purely mathematical properties of objects (such as having a well-defined notion of size/volume), and their logical description (which seems to be a human invention)?

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1. The world is inherently 'logical' in nature. Logically simpler objects are somehow 'nicer' in reality.
2. The concepts which humans devised to describe/model the world, are (subconsciously) based on some kind of language, and therefore are of limited logical complexity and/or are somehow related to logic.

What next?

$$
\begin{array}{ccccccccccccccc}
\text{(open)} & & & & & & & & & & \text{(analytic)} & & & & \\
\mathbf{\Sigma}^0_1 & & & & \mathbf{\Sigma}^0_2 & & & & & & \mathbf{\Sigma}^1_1 & & & & \mathbf{\Sigma}^1_2 \\
& \subseteq & & \subseteq & & \subseteq & & \text{(Borel)} & \subseteq & & & \subseteq & & \subseteq & & \subseteq \\
& & \mathbf{\Delta}^0_2 & & & & \cdots & \mathbf{\Delta}^1_1 & & & & & \mathbf{\Delta}^1_2 & & & & \cdots \\
& \subseteq & & \subseteq & & \subseteq & & & \subseteq & & & \subseteq & & \subseteq & & \subseteq \\
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OK!

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 & & \boldsymbol{\Delta}^0_2 & & & & \dots & \boldsymbol{\Delta}^1_1 & & & & & \boldsymbol{\Delta}^1_2 & & & \dots \\
 & \supseteq & & \subseteq & & \supseteq & & & \subseteq & & & \supseteq & \subseteq & & \supseteq & \\
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OK! ?

Are all projective sets regular?

What next?

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& & \mathbf{\Delta}^0_2 & & & & \cdots & \overset{\text{(Borel)}}{\mathbf{\Delta}^1_1} & & & \mathbf{\Delta}^1_2 & & & \cdots \\
& \supseteq & & \subseteq & & \supseteq & & \subseteq & & \supseteq & & \subseteq & & \supseteq \\
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OK! ?

Are all projective sets regular?

Many mathematicians worked on this problem in the early 20th century but were unable to solve it.

What next?

"Les efforts que j'ai faits pour résoudre cette question m'ont conduit à ce résultat tout inattendu: il existe une famille ... d'ensembles effectifs telle qu'on ne sait pas et l'on ne saura jamais *si un ensemble quelconque de cette famille (supposé non dénombrable) a la puissance du continu, s'il est ou non de troisième catégorie, ni même s'il est mesurable ... c'est la famille des* ensebles projectifs *de M. H. Lebesuge. Il ne reste donc qu'à reconnaître la nature de ce fait nouveau."* [Luzin, 1925]

"The efforts that I exerted on the resolution of this question led me to the following totally unexpected discovery: there exists a family ... consisting of effective [definable] sets, such that one does not know *and one will never know* whether every set from this family, if uncountable, has the cardinality of the continuum, nor whether it is of the third category, nor whether it is measurable. ... This is the family of the *projective sets* of Mr. H. Lebesgue. It remains but to recognize the nature of this new development."

3. Gödel, Solovay, Shelah

Constructible universe

Gödel's constructible universe:

- $L_0 := \varnothing$
- $L_{\alpha+1} := \mathrm{Def}(L_\alpha)$
- $L_\lambda := \bigcup_{\alpha<\lambda} L_\alpha$ (for limit ordinals λ).

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$\mathbb{L}$ is a so-called *inner model* of set theory: it satisfies all axioms of ZFC, plus additional axioms (e.g., GCH). The "real universe" $\mathbb{V}$ could be $\mathbb{L}$, or it could be larger than $\mathbb{L}$. The statement "$\mathbb{V} = \mathbb{L}$" is the *axiom of constructibiliy*.

Well-ordering the reals in $\mathbb{L}$

Fact (Gödel, 1938): in $\mathbb{L}$, there is a $\mathbf{\Sigma}^1_2$ well-ordering $<_{\mathbb{L}}$ of $\mathbb{R}$.

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Even more: there is a $\mathbf{\Sigma}^1_2$-*good* well-ordering of $\mathbb{R}$, meaning that the relation $\Phi(x, y)$ defined by

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Theorem (Gödel, 1938)

One cannot prove in ZFC *that all* $\mathbf{\Sigma}^1_2$ *sets are Lebesgue measurable or have the Property of Baire. One cannot prove in* ZFC *that all* $\mathbf{\Pi}^1_1$ *sets satisfy the Perfect Set Property.*

Forcing

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- In a specific way, from $\mathbb{P}$ one derives a so-called *generic object* G, which lies outside the model M.
- $M[G]$: least model of ZFC extending M and containing G.
- Using technical properties of $\mathbb{P}$, we have some control over the additional axioms that hold in $M[G]$.

Solovay's result

Theorem (Solovay, 1970)

There is a forcing extension $\mathbb{L}[G]$ of $\mathbb{L}$ in which all $\mathbf{\Sigma}^1_2$ sets are Lebesgue measurable and satisfy the Property of Baire.

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Corollary: the measurability (and PoB) of all $\mathbf{\Sigma}^1_2$ sets is independent of ZFC.

Other results of Solovay

Theorem (Solovay, 1970)

If M is a model of ZFC+ *"there exists an inaccessible cardinal", then there is a forcing extension $M[G]$ of M in which all* **projective** *sets are Lebesgue measurable, satisfy the Property of Baire and the Perfect Set Property.*

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Corollary: the measurability (and PoB and PSP) of all projective sets is independent of ZFC (plus inaccessible).

Even more results of Solovay

Theorem (Solovay, 1970)

Let M and $M[G]$ be as before. In $M[G]$, there is an inner model which satisfies ZF *but not* AC*, and in which* **all** *sets are measurable, satisfy the Property of Baire and the Perfect Set Property.*

Even more results of Solovay

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Corollary: the existence of non-measurable (non-PoB, non-PSP) sets cannot be proved without the Axiom of Choice!

Measurability and size of the universe

Recall:

- $\mathbb{L}$ is the smallest inner model. $\mathbb{V} = \mathbb{L}$ is a statement about the **minimality** of the universe.
- Using forcing, we can add generic object G to $\mathbb{L}$, producing a larger universe $\mathbb{V} = \mathbb{L}[G]$.
- One particular forcing: *random forcing* (due to [Solovay, 1970]).

Measurability and size of the universe

Theorem (Judah-Shelah, 1989)

The following are equivalent:

1. *all* $\mathbf{\Delta}^1_2$ *sets are Lebesgue-measurable,*
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In both cases, point 2 asserts "transcendence over $\mathbb{L}$" (i.e., in which way the actual universe is larger than the minimal one).

Other properties

- Given a regularity property, one hopes to find an equivalence theorem like the one above, but for different notions of "transcendence over $\mathbb{L}$".

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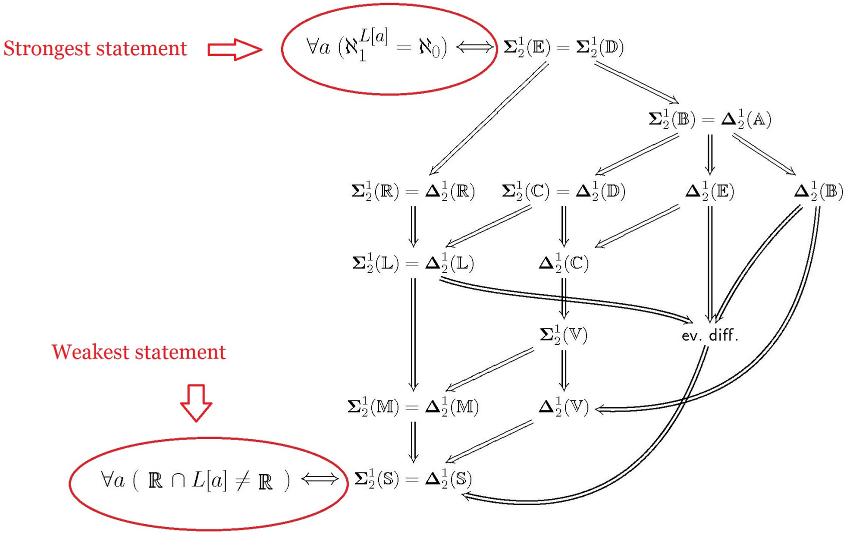
Other properties

- Given a regularity property, one hopes to find an equivalence theorem like the one above, but for different notions of "transcendence over $\mathbb{L}$".
- Various results proved by Judah, Shelah, Brendle, Löwe, Halbeisen, Ikegami.
- Transcendence can have different "strength", e.g.:
  - "all $\mathbf{\Sigma}^1_2$ sets are Marczewski-measurable" is equivalent to "$\forall r \in \mathbb{R}\ (\mathbb{R} \cap \mathbb{L}[r] \neq \mathbb{R})$",
  - "all $\mathbf{\Pi}^1_1$ sets satisfy the Perfect Set Property" is equivalen to "$\forall r \in \mathbb{R}\ (|\mathbb{R} \cap \mathbb{L}[r]| = \aleph_0)$".



Brendle & Löwe, *Eventually different functions and inaccessible cardinals.*

$$
\begin{tikzcd}
\boldsymbol{\Sigma}^1_2(\mathrm{Ramsey}) \arrow[dd, Rightarrow] \arrow[dr, Rightarrow, "(a)"] & & \boldsymbol{\Sigma}^1_2(\vec{n} \to \vec{m}) \arrow[dl, Rightarrow, "(b)"'] \arrow[d, Rightarrow, "(c')"] & \\
& \boldsymbol{\Sigma}^1_2(\vec{\omega} \to \vec{m}) \arrow[d, Rightarrow, "(c)"'] \arrow[ddr, Rightarrow, bend left, "(e')"] & \boldsymbol{\Delta}^1_2(\vec{n} \to \vec{m}) \arrow[dl, Rightarrow, "(b')"'] \arrow[d, Rightarrow, "(d')"] & \\
\begin{array}{c}\boldsymbol{\Sigma}^1_2(\mathbb{L}) \\ \boldsymbol{\Delta}^1_2(\mathbb{L})\end{array} \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \boldsymbol{\Delta}^1_2(\vec{\omega} \to \vec{m}) \arrow[d, Rightarrow, "(d)"'] \arrow[ddrr, Rightarrow, bend left, "(e)"] & \text{bd. ev. diff.} \arrow[dl, Rightarrow, "(b'')"'] & \boldsymbol{\Sigma}^1_2(\mathrm{doughnut}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \\
\begin{array}{c}\boldsymbol{\Sigma}^1_2(\mathbb{M}) \\ \boldsymbol{\Delta}^1_2(\mathbb{M})\end{array} & \text{ev. diff.} & \boldsymbol{\Sigma}^1_2(\mathbb{E}_0) \arrow[d, Rightarrow] & \boldsymbol{\Delta}^1_2(\mathrm{doughnut}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \\
& & \boldsymbol{\Delta}^1_2(\mathbb{E}_0) & \boldsymbol{\Delta}^1_2(\mathbb{SPL})
\end{tikzcd}
$$

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4. Maximal almost disjoint (m.a.d.) families.

Thank you!

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