Cichoń's Diagram and Regularity Properties

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Cichoń's diagram

"What pentagram is to heavy metal, Cichoń's diagam is to set theory."

— Jindřich Zapletal

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 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
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\end{tikzcd}
$$

1. Each inequality appearing in the diagram is provable in ZFC.
2. Each inequality **not** appearing in the diagram is **not** provable in ZFC, except
3. $\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$ and $\mathrm{cof}(\mathcal{M}) = \max(\mathfrak{d}, \mathrm{non}(\mathcal{M}))$.

Regularity properties

Let A be a set of reals (ω^ω or 2^ω).

- A has the **Baire property** iff for every basic open $[s]$ there is a basic open $[t] \subseteq [s]$ such that $[t] \subseteq^* A$ or $[t] \cap A =^* \varnothing$.

  ($\subseteq^*$ and $=^*$ means **modulo meager**)

- A is **Lebesgue-measurable** iff for every closed set C of positive measure there is a closed subset $C' \subseteq C$ of positive measure, such that $C \subseteq A$ or $C \cap A = \varnothing$.

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Baire property = Cohen forcing

Lebesgue measure = random forcing

More forcing

Sacks forcing $\mathbb{S}$: **perfect trees** on $2^{<\omega}$ ordered by inclusion.

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Set theorists get born and die, move to distant countries, get married, bear children, go bankrupt, grow old and sick, and Sacks forcing is still with us, working just as well as the day Sacks invented it.

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Even more forcing

Miller forcing $\mathbb{M}$: **super-perfect trees** on $\omega^{<\omega}$ ordered by inclusion.

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(A tree is **super-perfect** if every node has an extensions which is infinitely splitting).

Laver forcing $\mathbb{L}$: **Laver trees** on $\omega^{<\omega}$ ordered by inclusion.

(A tree is **Laver** if every node beyond the stem is infinitely splitting).

More Regularity Properties

Definition

$A \subseteq 2^\omega$ is **Sacks-measurable** (Marczewski-measurable) iff

$$\forall T \in \mathbb{S}\, \exists S \leq T\ ([S] \subseteq A \text{ or } [S] \cap A = \varnothing).$$

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$$\forall T \in \mathbb{M}\, \exists S \leq T\ ([S] \subseteq A \text{ or } [S] \cap A = \varnothing).$$

$A \subseteq \omega^\omega$ is **Laver-measurable** iff

$$\forall T \in \mathbb{L}\, \exists S \leq T\ ([S] \subseteq A \text{ or } [S] \cap A = \varnothing).$$

Applications

Baire and Lebesgue properties have many applications (topology, analysis etc.) The other (Marczewski-style) properties were introduced by Polish mathematicians in the 1930s, and have several applications in

- Forcing theory
- topology
- infinitary combinatorics
- Ramsey theory
- etc.

Polish mathematicians were already interested in them in the 1930s.

Projective sets

Let $\mathbf{\Gamma}$ be a projective pointclass (e.g., Borel, $\mathbf{\Sigma}^1_1$, Projective etc.)

"$\mathbf{\Gamma}(\mathbb{P})$" abbreviates the statement "all sets of complexity $\mathbf{\Gamma}$ are $\mathbb{P}$-measurable".

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$\mathbf{\Sigma}^1_1(\mathbb{P})$ is true (for all $\mathbb{P}$)

But $\mathbf{\Sigma}^1_2(\mathbb{P})$ and $\mathbf{\Delta}^1_2(\mathbb{P})$ are independent of ZFC.

Cichoń's diagram for regularity properties

$$
\begin{tikzcd}
 & \boldsymbol{\Delta}^1_2(\mathbb{B}) \arrow[rrr, Rightarrow] & & & \begin{array}{c}\boldsymbol{\Sigma}^1_2(\mathbb{S})\\ \boldsymbol{\Delta}^1_2(\mathbb{S})\end{array} \\
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1. Each implication appearing in the diagram is provable in ZFC.
2. Each implication **not** appearing in the diagram is **not** provable in ZFC, except
3. $\mathbf{\Delta}^1_2(\mathbb{L}) + \mathbf{\Delta}^1_2(\mathbb{C}) \implies \mathbf{\Sigma}^1_2(\mathbb{C})$

Characterization (1)

Why the analogy with cardinal invariants?

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Theorem (Judah-Shelah 1989)

The following are equivalent:

1. $\mathbf{\Delta}^1_2(\mathbb{C})$
2. $\forall r\, \exists x\, (x \text{ is Cohen over } L[r])$.

Theorem (Solovay 1970)

The following are equivalent:

1. $\mathbf{\Sigma}^1_2(\mathbb{C})$
2. $\forall r\, \{x \mid x \text{ Cohen over } L[r]\}$ *is comeager.*

Characterization (2)

Why this analogy?

Theorem (Judah-Shelah 1989)

The following are equivalent:

1. $\mathbf{\Delta}^1_2(\mathbb{B})$
2. $\forall r\, \exists x\, (x$ *is random over* $L[r])$.

Theorem (Solovay 1970)

The following are equivalent:

1. $\mathbf{\Sigma}^1_2(\mathbb{B})$
2. $\forall r\, \mu(\{x \mid x$ *random over* $L[r]\}) = 1$.

Characterization (3)

Theorem (Brendle-Löwe 1999)

- *The following are equivalent:*
  1. $\mathbf{\Delta}^1_2(\mathbb{L})$
  2. $\mathbf{\Sigma}^1_2(\mathbb{L})$
  3. $\forall r\, \exists x\, (x$ *is dominating over* $L[r])$
- *The following are equivalent:*
  1. $\mathbf{\Delta}^1_2(\mathbb{M})$
  2. $\mathbf{\Sigma}^1_2(\mathbb{M})$
  3. $\forall r\, \exists x\, (x$ *is unbounded over* $L[r])$
- *The following are equivalent:*
  1. $\mathbf{\Delta}^1_2(\mathbb{S})$
  2. $\mathbf{\Sigma}^1_2(\mathbb{S})$
  3. $\forall r\, \exists x\, (x \notin L[r])$

Correspondence

Regularity hypothesis	Transcendence over $L[r]$	Cardinal characteristic
$\forall r(\omega_1^{L[r]} < \omega_1)$	"making ground-model reals countable"	$\aleph_1$
$\boldsymbol{\Sigma}^1_2(\mathbb{B})$	measure-one many random reals	$\mathrm{add}(\mathcal{N})$
$\boldsymbol{\Delta}^1_2(\mathbb{B})$	random reals	$\mathrm{cov}(\mathcal{N})$
$\boldsymbol{\Sigma}^1_2(\mathbb{C})$	co-meager many Cohen reals	$\mathrm{add}(\mathcal{M})$
$\boldsymbol{\Delta}^1_2(\mathbb{C})$	Cohen reals	$\mathrm{cov}(\mathcal{M})$
$\boldsymbol{\Delta}^1_2(\mathbb{L})$ / $\boldsymbol{\Sigma}^1_2(\mathbb{L})$	dominating reals	$\mathfrak{b}$
$\boldsymbol{\Delta}^1_2(\mathbb{M})$ / $\boldsymbol{\Sigma}^1_2(\mathbb{M})$	unbounded reals	$\mathfrak{d}$
$\boldsymbol{\Delta}^1_2(\mathbb{S})$ / $\boldsymbol{\Sigma}^1_2(\mathbb{S})$	new reals	$2^{\aleph_0}$

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Analogy between hypotheses about regularity on 2nd level and cardinal invariants.

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Analogy between hypotheses about regularity on 2nd level and cardinal invariants.

Question

What happens at higher levels of the projective hierarchy?

Some straightforward implications

Note that **some** of the implications are straightforward.

Fact

Let Γ be **any** pointclass closed under continuous pre-images. Then:

1. $\Gamma(\mathbb{L}) \Rightarrow \Gamma(\mathbb{M}) \Rightarrow \Gamma(\mathbb{S})$.
2. $\Gamma(\mathbb{B}) \Rightarrow \Gamma(\mathbb{S})$.
3. $\Gamma(\mathbb{C}) \Rightarrow \Gamma(\mathbb{M})$.

Cichoń's diagram on the third level

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Eventually, we would like to "solve" this diagram in ZFC or ZFC + inaccessible.

Cohen and Random

Concerning Cohen and Random, some things were known:

- Con($\mathbf{\Delta}^1_3(\mathbb{C}) + \neg\mathbf{\Delta}^1_3(\mathbb{B})$) from ZFC (Bagaria-Judah 1993)

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- Con($\mathbf{Proj}(\mathbb{B}) + \neg\mathbf{\Delta}^1_3(\mathbb{C})$) from ZFC + Mahlo (Friedman-Schrittesser 2013).

Solving the diagrams

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Are $\mathbf{\Sigma}^1_3(\mathbb{P})$ and $\mathbf{\Delta}^1_3(\mathbb{P})$ equivalent for $\mathbb{P} \in \{\mathbb{S}, \mathbb{L}, \mathbb{M}\}$?

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But it is easier if we restrict attention exclusively to $\mathbf{\Delta}^1_3$, $\mathbf{\Sigma}^1_3$ or $\mathbf{\Delta}^1_4$ sets!

The Γ-diagram

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We will try to "solve" this diagram for $\boldsymbol{\Gamma} = \boldsymbol{\Delta}^1_3$, $\boldsymbol{\Sigma}^1_3$ and $\boldsymbol{\Delta}^1_4$.

The $\mathbf{\Gamma}$-diagram

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We will try to "solve" this diagram for $\mathbf{\Gamma} = \mathbf{\Delta}^1_3$, $\mathbf{\Sigma}^1_3$ and $\mathbf{\Delta}^1_4$.

Main difficulty: develop methods to obtain $\mathbf{\Gamma}(\mathbb{P})$ by forcing iterations, without too much damage.

Classical results

Classical methods (Solovay, Judah-Shelah) essentially say:

Fact

1. An iteration of length ω_1 of $\mathbb{P}$ yields $\mathbf{\Delta}^1_2(\mathbb{P})$, and
2. An iteration of length ω_1 of "amoeba-for-$\mathbb{P}$" yields $\mathbf{\Sigma}^1_2(\mathbb{P})$.

Suslin$^+$ proper forcing

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By "simple definitions" we mean that the conditions are (coded by) reals and

1. **Suslin:** "$p \in \mathbb{P}$","$p \leq q$" and "$p \perp q$" are $\boldsymbol{\Sigma}^1_1$ relations.
2. **Suslin$^+$:** "$p \in \mathbb{P}$" and "$p \leq q$" are $\boldsymbol{\Sigma}^1_1$, and "being pre-dense below a condition" can, w.l.o.g., be stated in a $\boldsymbol{\Sigma}^1_2$-way.

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All standard definable forcings used in the theory of the reals which are known to be proper, are actually Suslin$^+$ proper.

Amoeba and Quasi-amoeba

Definition

Let $\mathbb{P}$ be a tree-like forcing notion, and $\mathbb{AP}$ another forcing. We say that

1. $\mathbb{AP}$ is a **quasi-amoeba for** $\mathbb{P}$ if for every $p \in \mathbb{P}$ and every $\mathbb{AP}$-generic G, there is a $q \in \mathbb{P}^{V[G]}$ such that $V[G] \models q \leq_{\mathbb{P}} p$ and

$$V[G] \models \forall x \in [q]\,(x \text{ is } \mathbb{P}\text{-generic over } V).$$

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$$W \models \forall x \in [q]\,(x \text{ is } \mathbb{P}\text{-generic over } V).$$

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2. $\mathbb{AP}$ is an **amoeba for** $\mathbb{P}$ if for every $p \in \mathbb{P}$ and every $\mathbb{AP}$-generic G, there is a $q \in \mathbb{P}^{V[G]}$ such that $V[G] \models q \leq_{\mathbb{P}} p$ and **for any larger model** $W \supseteq V[G]$,
$$W \models \forall x \in [q]\,(x \text{ is } \mathbb{P}\text{-generic over } V).$$

Examples:

1. $\mathbb{S}$ is a quasi-amoeba, but not an amoeba, for itself (Brendle 1998).
2. $\mathbb{M}$ is a quasi-amoeba, but not an amoeba, for itself (Brendle 1998).

The methods

Method 1 (Bagaria-Judah)

1. If $V \models \mathbf{\Sigma}^1_2(\mathbb{B})$ then $V^{\mathbb{B}_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{B})$.
2. If $V \models \mathbf{\Sigma}^1_2(\mathbb{C})$ then $V^{\mathbb{C}_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{C})$.

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Method 2 (Fischer-Friedman-Kh)

Suppose $\mathbb{AP}_i$ is a quasi-amoeba for $\mathbb{P}_i$ for all $i \leq k$, and all $\mathbb{P}_i$ and $\mathbb{AP}_i$ are Suslin$^+$ proper. Then $V^{(\mathbb{P}_0 * \mathbb{AP}_0 * \cdots * \mathbb{P}_k * \mathbb{AP}_k)_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{P}_i)$ for each i.

More methods

Method 3 (Fischer-Friedman-Kh)

Suppose $V \models \forall r\,(\omega_1^{L[r]} < \omega_1)$ and $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ is an iteration of Suslin$^+$ proper forcing notions in which $\mathbb{P}$ appears cofinally often. Then $V^{\mathbb{P}_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{P})$.

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Separation

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We almost have all ingredients necessary to separate regularity properties. We need to fix the appropriate ground model:

1. To solve the $\mathbf{\Sigma}^1_2/\mathbf{\Delta}^1_2$-diagram, L was used as the ground model, because L has a Σ^1_2-**good wellorder of the reals.**
2. (Bagaria-Woodin; Friedman) From ZFC, we can construct a model L^* in which
   - $\mathbf{\Sigma}^1_2(\mathbb{P})$ holds for all $\mathbb{P}$, but
   - there is a Σ^1_3-good wellorder of the reals.
3. (Rene David) From ZFC + inaccessible, we can construct a model L^d in which
   - $\forall r\, (\omega_1^{L[r]} < \omega_1)$, but
   - there is a Σ^1_3-good wellorder of the reals.

$$
\begin{tikzcd}
\boldsymbol{\Delta}^1_3(\mathbb{B}) \arrow[rrr, Rightarrow] & & & \boldsymbol{\Delta}^1_3(\mathbb{S}) \\
& \boldsymbol{\Delta}^1_3(\mathbb{L}) \arrow[r, Rightarrow] & \boldsymbol{\Delta}^1_3(\mathbb{M}) \arrow[ur, Rightarrow] & \\
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\end{tikzcd}
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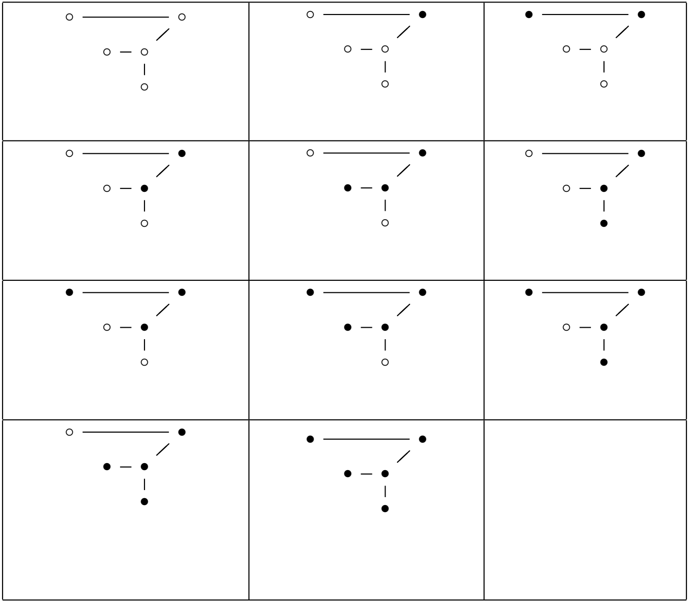

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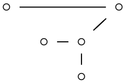
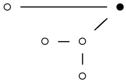
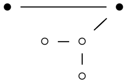
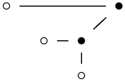
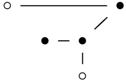
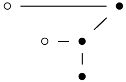
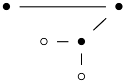
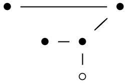
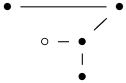
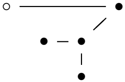
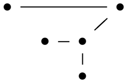
 L	 $L^{\mathbb{S}_{\omega_1}}$	 $(L^*)^{\mathbb{B}_{\omega_1}}$
 $L^{\mathbb{M}_{\omega_1}}$	 $L^{(\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $L^{\mathbb{R}_{\omega_1}}$	 $(L^*)^{\mathbb{C}_{\omega_1}}$
 $(L^d)^{(\mathbb{B}*\mathbb{M})_{\omega_1}}$	 $(L^d)^{(\mathbb{B}*\mathbb{L})_{\omega_1}}$	 $(L^d)^{(\mathbb{B}*\mathbb{C})_{\omega_1}}$
 $(L^d)^{(\mathbb{C}*\mathbb{L})_{\omega_1}}$ or a ZFC-model of Bartoszyński-Judah	 $L^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $L^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{R})_{\omega_1}}$	

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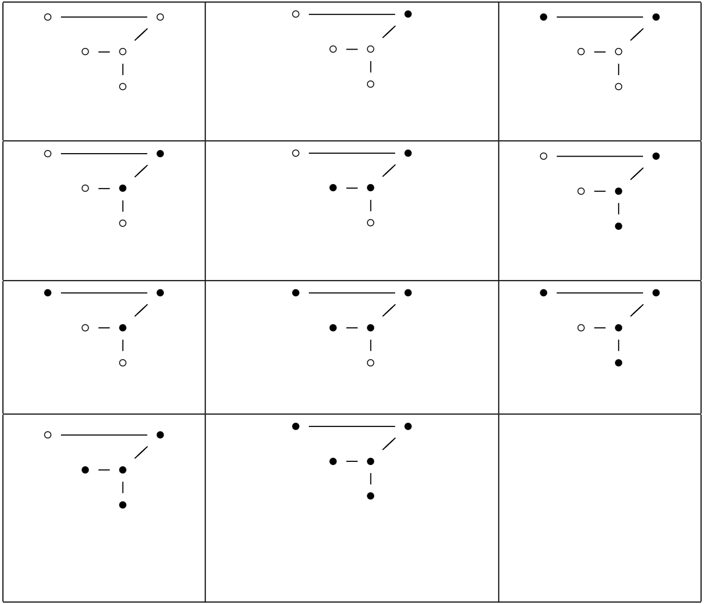

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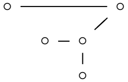
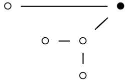
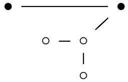
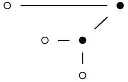
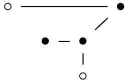
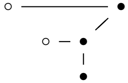
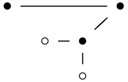
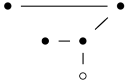
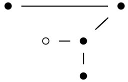
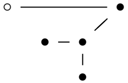
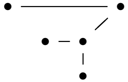
 L	 $(L^d)^{\mathbb{S}_{\omega_1}}$	 ???
 $(L^d)^{\mathbb{M}_{\omega_1}}$	 $(L^d)^{(\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $(L^d)^{\mathbb{R}_{\omega_1}}$	 ???
 ???	 ???	 ???
 ???	 $(L^d)^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $(L^d)^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{R})_{\omega_1}}$ (or Solovay Model)	

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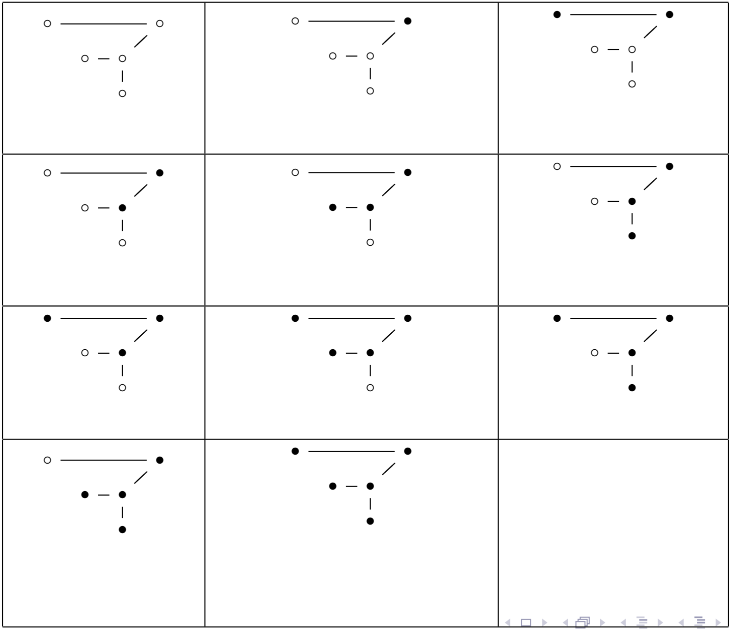

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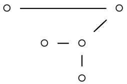
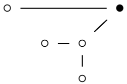
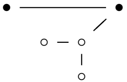
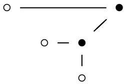
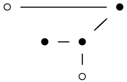
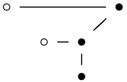
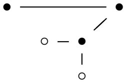
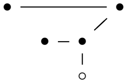
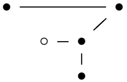
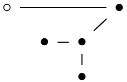
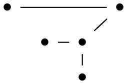
 L	 $(L^d)^{\mathbb{S}_{\omega_1}}$	 ???
 $(L^d)^{\mathbb{M}_{\omega_1}}$	 $(L^d)^{(\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $(L^d)^{\mathbb{R}_{\omega_1}}$	 ???
 ???	 ???	 ???
 ???	 $(L^d)^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $(L^d)^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{R})_{\omega_1}}$ (or Solovay Model)	

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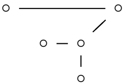
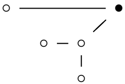
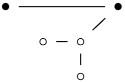
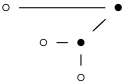
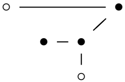
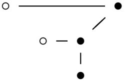
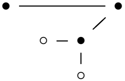
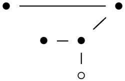
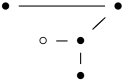
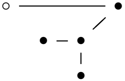
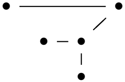
 L	 $(L^d)^{\mathbb{S}_{\omega_1}}$	 Judah-Spinas 1995
 $(L^d)^{\mathbb{M}_{\omega_1}}$	 $(L^d)^{(\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $(L^d)^{\mathbb{R}_{\omega_1}}$	 Judah-Spinas 1995
 ???	 ???	 ???
 ???	 $(L^d)^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $(L^d)^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{R})_{\omega_1}}$ (or Solovay Model)	

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3. Solve the other diagrams.

Open Questions

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1. Is $\mathbf{\Sigma}^1_3(\mathbb{P})$ and $\mathbf{\Delta}^1_3(\mathbb{P})$ equivalent for Sacks, Miller and Laver? (we conjecture that they are not).
2. Solve the $\mathbf{\Delta}^1_3$-diagram in ZFC.
3. Solve the other diagrams.
4. Consistency strength of $\mathbf{\Sigma}^1_3(\mathbb{L})$?

Thank you!

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