Polarized partition properties on the second level of the projective hierarchy

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Introduction: Regularity properties and the projective hierarchy.

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(e.g. Lebesgue measurability, Baire property, Ramsey property, Bernstein property)

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More regularity on $\mathbf{\Delta}^1_2/\mathbf{\Sigma}^1_2$-level $\quad \sim \quad$ *L gets smaller*

Examples

- $\mathbf{\Delta}^1_2(\text{Lebesgue}) \iff \forall a\ \exists$ random-generic$/L[a]$
- $\mathbf{\Delta}^1_2(\text{Baire Property}) \iff \forall a\ \exists$ Cohen-generic$/L[a]$

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- $\mathbf{\Delta}^1_2$(Lebesgue) $\iff$ $\forall a\ \exists$ random-generic$/L[a]$
- $\mathbf{\Delta}^1_2$(Baire Property) $\iff$ $\forall a\ \exists$ Cohen-generic$/L[a]$
- $\mathbf{\Delta}^1_2$(Ramsey) $\iff$ $\forall a\ \exists$ Ramsey real $/L[a]$
- $\mathbf{\Delta}^1_2$(Laver) $\iff$ $\forall a\ \exists$ dominating real $/L[a]$
- $\mathbf{\Delta}^1_2$(Miller) $\iff$ $\forall a\ \exists$ unbounded real $/L[a]$
- $\mathbf{\Delta}^1_2$(Sacks) $\iff$ $\forall a\ \exists$ real $\notin L[a]$

More Examples

- $\mathbf{\Sigma}^1_2$(Lebesgue) $\iff$ $\forall a\, \exists$ measure-one set of random-generics$/L[a]$
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Implications and non-implications

Given two regularity properties Reg_1 and Reg_2 we are interested in:

$$\mathbf{\Gamma}(\mathrm{Reg}_1) \implies \mathbf{\Gamma}'(\mathrm{Reg}_2)?$$

for $\mathbf{\Gamma}, \mathbf{\Gamma}' \in \{\mathbf{\Delta}^1_2, \mathbf{\Sigma}^1_2\}$

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What has been established so far?

Diagram of implications

$$
\begin{tikzcd}[column sep=small]
 & \mathbf{\Sigma}^1_2(\mathbb{E}) = \mathbf{\Sigma}^1_2(\mathbb{D}) \arrow[dl, Rightarrow, shift left=0] \arrow[ddl, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & & \mathbf{\Sigma}^1_2(\mathbb{B}) = \mathbf{\Delta}^1_2(\mathbb{A}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\
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\mathbf{\Sigma}^1_2(\mathbb{L}) = \mathbf{\Delta}^1_2(\mathbb{L}) \arrow[dd, Rightarrow] \arrow[drr, Rightarrow] & \mathbf{\Delta}^1_2(\mathbb{C}) \arrow[d, Rightarrow] & & \\
 & \mathbf{\Sigma}^1_2(\mathbb{V}) \arrow[dl, Rightarrow] \arrow[d, Rightarrow] & \text{ev. diff.} \arrow[ddll, Rightarrow, bend left] & \\
\mathbf{\Sigma}^1_2(\mathbb{M}) = \mathbf{\Delta}^1_2(\mathbb{M}) \arrow[d, Rightarrow] & \mathbf{\Delta}^1_2(\mathbb{V}) \arrow[dl, Rightarrow] & & \\
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\end{tikzcd}
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Diagram: Brendle & Löwe, *Eventually different functions and inaccessible cardinals*

Polarized partition properties.

Polarized partitions

We work in ω^ω. Letters $H, J, \ldots$ stand for infinite sequences of finite subsets of ω, i.e. $H : \omega \longrightarrow [\omega]^{<\omega}$. Use abbreviation: $[H] = \prod_{i \in \omega} H(i)$.

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Definition (unbounded polarized partition)

A set $A \subseteq \omega^\omega$ satisfies the property $\begin{pmatrix} \omega \\ \omega \\ \cdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \cdots \end{pmatrix}$ if

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and $n_1, n_2, \ldots$ are recursive in $m_1, m_2, \ldots$.

Some facts

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3. - $\mathbf{\Gamma}(\vec{\omega} \to \vec{m}) \iff \mathbf{\Gamma}(\vec{\omega} \to \vec{m'})$, for all $\vec{m}, \vec{m}' \geq 2$.
   - If $\mathbf{\Gamma}(\vec{n} \to \vec{m})$, then for every other $\vec{m'}$ there is $\vec{n'}$ such that $\mathbf{\Gamma}(\vec{n'} \to \vec{m'})$

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From now on, use generic notations $(\vec{\omega} \to \vec{m})$ and $(\vec{n} \to \vec{m})$.

Which sets satisfy this property?

In [DiPrisco & Todorčević, 2003]:

- $(\vec{\omega} \to \vec{m})$ and $(\vec{n} \to \vec{m})$ hold for analytic sets.
- Explicit bounds $\vec{n}$ computed from $\vec{m}$ (using Ackermann-like function).

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So, what about $\mathbf{\Delta}^1_2/\mathbf{\Sigma}^1_2(\vec{\omega} \to \vec{m})$ and $\mathbf{\Delta}^1_2/\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$?

$\mathbf{\Delta^1_2}$-level: easy results.

Upper and lower bounds

Theorem

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Theorem (Brendle)

If $\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m})$ *then* $\forall a$ *there is an eventually different real over* $L[a]$.

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If $\mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m})$ then $\forall a$ there is an eventually different real over $L[a]$.

Proof.

Assume otherwise and use the canonical $\Delta^1_2(a)$-well-ordering of $L[a]$ to construct a counterexample. □

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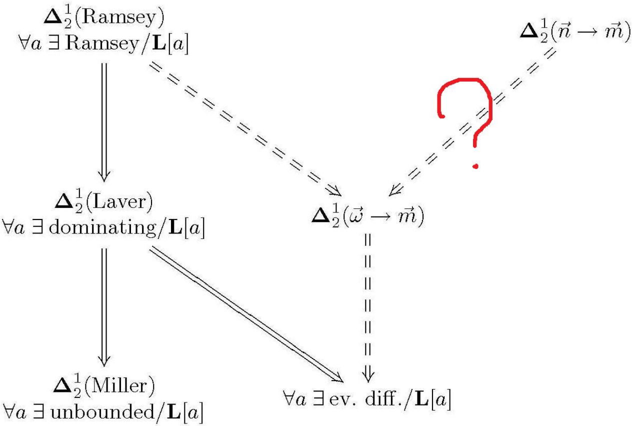

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Theorem (Brendle-Kh)

Let V be obtained by an ω_1-iteration of Mathias forcing beginning from L. Then $\mathbf{\Delta}^1_2(Ramsey)$ holds whereas $\mathbf{\Delta}^1_2(\vec{n} \rightarrow \vec{m})$ fails.

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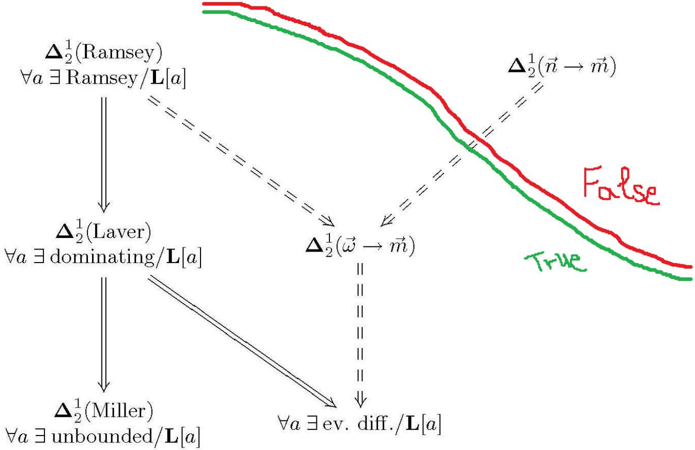

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Proof.

Use the fact that Mathias forcing satisfies the **Laver property**. □

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$\mathbf{\Delta}^1_2$-level: creature forcing.

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There is a model in which $\mathbf{\Delta}^1_2(\vec{n} \to \vec{m})$ holds but $\mathbf{\Delta}^1_2(Miller)$ fails.
(i.e. there are **no unbounded reals***).*

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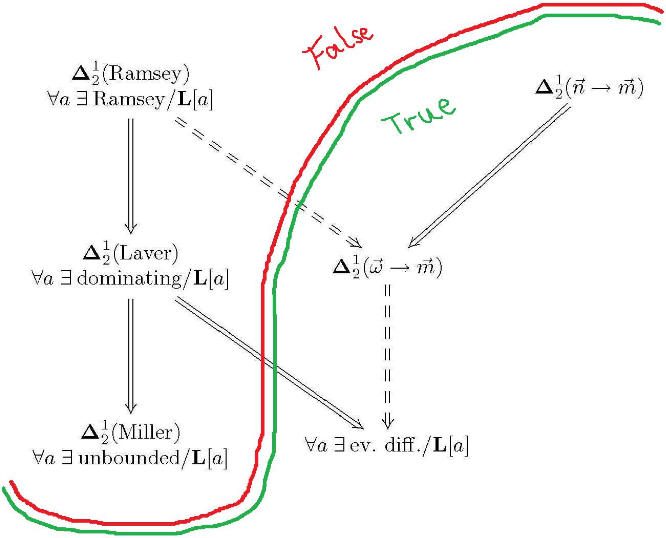

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    - $(c', k') \leq_n (c, k)$ iff $c' \subseteq c$ and $k' \geq k$.
- Let $a_n := 2^{1/\epsilon_n}$. Define $\mathrm{norm}_n(c, k) := \log_{a_n}(\log_3(|c|) - k)$ for each $(c, k) \in \mathbb{P}_n$.

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- Let $a_n := 2^{1/\epsilon_n}$. Define $\mathrm{norm}_n(c, k) := \log_{a_n}(\log_3(|c|) - k)$ for each $(c, k) \in \mathbb{P}_n$.

If $X(n)$ is sufficiently large, then $\exists(c, k) \in \mathbb{P}_n$ s.t. $\mathrm{norm}_n(c, k) \geq n$.

[To be precise: $X(n)$ must be larger then $2^{((2^{1/\epsilon_n})^n)}$]

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Proper and ω^ω-bounding

Theorem (Kellner-Shelah, Shelah-Zapletal)

If $\mathbb{P}_{\mathrm{KSZ}}$ is as above, and moreover $\forall n \left(\epsilon_n \leq \frac{1}{n \cdot \prod_{j<n} X(j)}\right)$, then $\mathbb{P}_{\mathrm{KSZ}}$ is proper and ω^ω-bounding.

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Show that each component $\mathbb{P}_n$ of $\mathbb{P}_{\mathrm{KSZ}}$ satisfies two properties from the general theory of creature forcings: "ϵ_n-*bigness*" and "ϵ_n-*halving*". $\square$

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Remark: $X(n)$ is a function of ϵ_n, and ϵ_n is a function of $X(m)$ for $m < n$. So we have to define them inductively.

Forcing $\boldsymbol{\Delta}^1_2(\vec{n} \to \vec{m})$

Theorem (Brendle-Kh)

An ω_1-iteration of $\mathbb{P}_{\mathrm{KSZ}}$, starting from L, gives a model in which $\boldsymbol{\Delta}^1_2(\vec{n} \to \vec{m})$ holds but $\boldsymbol{\Delta}^1_2(\mathit{Miller})$ fails.

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The bounds "$\vec{n}$" have been explicitly computed beforehand: they are the $X(n)$'s from the definition of $\mathbb{P}_{\mathrm{KSZ}}$.

Diagram of implications

$$
\begin{tikzcd}
\begin{array}{c}\mathbf{\Delta}^1_2(\text{Ramsey}) \\ \forall a\, \exists\, \text{Ramsey}/\mathbf{L}[a]\end{array} \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & & \mathbf{\Delta}^1_2(\vec{n} \to \vec{m}) \arrow[dl, Rightarrow] \\
\begin{array}{c}\mathbf{\Delta}^1_2(\text{Laver}) \\ \forall a\, \exists\, \text{dominating}/\mathbf{L}[a]\end{array} \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \mathbf{\Delta}^1_2(\vec{\omega} \to \vec{m}) \arrow[d, Rightarrow, dashed] & \\
\begin{array}{c}\mathbf{\Delta}^1_2(\text{Miller}) \\ \forall a\, \exists\, \text{unbounded}/\mathbf{L}[a]\end{array} & \forall a\, \exists\, \text{ev. diff.}/\mathbf{L}[a] &
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4. Same for $(\vec{n} \to \vec{m})$.

The $\mathbf{\Sigma}^1_2$-level

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An ω_1-iteration of any (proper) forcing notion with the clopification property, starting from L, gives a model where $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ holds.

Forcing $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$: continued

Problem with using the DiPrisco-Todorčević forcing: difficult to see whether it is ω^ω-bounding or not.

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Corollary

There is a model where $\mathbf{\Sigma}^1_2(\vec{n} \to \vec{m})$ holds but $\mathbf{\Sigma}^1_2(Miller)$ fails.

Thank you!

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- Carlos A. Di Prisco, Stevo Todorčević, *Souslin partitions of products of finite sets*, Advances in Mathematics, vol. 176 (2003), pp. 145–173.
- Jakob Kellner, Saharon Shelah, *Decisive creatures and large continuum*, Journal of Symbolic Logic, Volume 74, Issue 1 (2009), pp 73–104; [KrSh:872].
- Andrzej Rosłanowski, Saharon Shelah, *Norms on Possibilities I: Forcing with Trees and Creatures*, Memoirs of the American Mathematical Society 671 (1999); [RoSh:470].