MAD families and the projective hierarchy

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If $V = L$ *then there is a* $\boldsymbol{\Sigma}^1_2$ *MAD family* (use the $\boldsymbol{\Sigma}^1_2$ well-ordering of $[\omega]^\omega$).

Theorem (Miller 1989)

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Definition

Let $\mathbb{P}$ be a (proper) forcing. A MAD family $\mathcal{A}$ is called $\mathbb{P}$*-indestructible* if it remains a MAD family in $V^{\mathbb{P}}$.

Preservation of $\mathbf{\Sigma}^1_2$ definition

Fact

If $\mathcal{A} \in \mathrm{L}$ is a $\mathbf{\Sigma}^1_2$ $\mathbb{P}$-indestructible MAD family, then in $\mathrm{L}^{\mathbb{P}}$ it is still $\mathbf{\Sigma}^1_2$.

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Proof.

If $\phi(x)$ is $\mathbf{\Sigma}^1_2$ and defines $\mathcal{A}$ in L, then "$\phi(x) \wedge x \in \mathsf{L}$" is also $\mathbf{\Sigma}^1_2$ and defines $\mathcal{A}$ in $\mathsf{L}^{\mathbb{P}}$. □

Preservation of $\mathbf{\Pi}^1_1$ definition

Fact (Friedman & Zdomskyy 2010)

If $\mathcal{A} \in \mathsf{L}$ is a $\mathbf{\Pi}^1_1$ $\mathbb{P}$-indestructible MAD family, then in $\mathsf{L}^{\mathbb{P}}$ it is still $\mathbf{\Pi}^1_1$.

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Proof.

Let $\phi(x)$ define $\mathcal{A}$ in L, then in $\mathsf{L}^{\mathbb{P}}$ it defines a larger family $\mathcal{A}'$. But the statement "$\forall x \forall y\ (\phi(x) \wedge \phi(y) \rightarrow x \cap y$ is finite)" has complexity $\mathbf{\Pi}^1_2$ and holds in L, so by Shoenfield absoluteness, it holds in $\mathsf{L}^{\mathbb{P}}$. Therefore $\mathcal{A}'$ is a.d., but since $\mathcal{A} \subseteq \mathcal{A}'$ and $\mathcal{A}$ is maximal, it must be the case that $\mathcal{A} = \mathcal{A}'$. Therefore $\mathcal{A}$ has a $\mathbf{\Pi}^1_1$ definition. □

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Corollary

CON($\neg$CH + $\exists \mathbf{\Pi}^1_1$ *MAD*).

Dominating reals

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Questions:

- Is "$\mathfrak{b} > \aleph_1 + \exists \mathbf{\Sigma}^1_2$ MAD" consistent?
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- To build a model of "$\mathfrak{b} > \aleph_1 + \exists \mathbf{\Sigma}^1_2$ MAD", previous methods don't suffice, because they only produce MAD families $\mathcal{A} \subseteq L$.
- To avoid this problem, we consider MAD families defined by $\aleph_1$-unions of perfect sets.

$\aleph_1$-perfect MAD

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2. An $\aleph_1$-perfect MAD $\mathcal{A}$ is $\mathbb{P}$-*indestructible* if in $\mathsf{V}^{\mathbb{P}}$, $\mathcal{A}^{\mathsf{V}^{\mathbb{P}}} := \bigcup\{P_\alpha^{\mathsf{V}^{\mathbb{P}}} \mid \alpha < \aleph_1\}$ is MAD.

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NB. If $\mathbb{P}$ adds a dominating real, it will destroy the $\aleph_1$-union of the *old* perfect sets, but not necessarily that of the *new* perfect sets.

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Recall:

- Hechler forcing $\mathbb{D}$ consist of conditions $(s, f) \in \omega^{<\omega} \times \omega^\omega$ with $s \subseteq f$, ordered by

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- $\mathbb{D}$ *preserves splitting families*: if $\mathcal{S} \subseteq [\omega]^\omega$ is a splitting family in V then it is still a splitting family in $V^{\mathbb{D}}$.

To prove the theorem, it suffices to construct a $\mathbb{D}$-indestructible, $\mathbf{\Sigma}^1_2$-definable, $\aleph_1$-perfect MAD family in L.

Proof

Proof: By simultaneous induction, construct a sequence $\langle P_\alpha \mid \alpha < \omega_1 \rangle$ of perfect a.d. sets, and an increasing sequence $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ of countable models, such that the following conditions are satisfied:

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(4) For all α and all $Y \in [\omega]^\omega \cap M_\alpha$: if Y is a.d. from $\bigcup_{\xi \leq \alpha} P_\xi$, then it is *not* a.d. from $P_{\alpha+1}$.

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(4′) Let G be generic for the $\aleph_2$-iteration of $\mathbb{D}$.
For all α and all $Y \in [\omega]^\omega \cap M_\alpha[G]$: if Y is a.d. from $\bigcup_{\xi\leq\alpha} P_\xi^{V[G]}$, then it is *not* a.d. from $P_{\alpha+1}^{V[G]}$.

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    - First attempt: let α be such that $\dot{Y} \in M_\alpha$. Then $Y \in M_\alpha[G]$, so apply condition (4′) and we are done.

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    Problem: $\bigcup_{\alpha<\omega_1} M_\alpha$ cannot cover all names for reals, because we are dealing with an $\aleph_2$-iteration.

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- In the $\aleph_2$-iteration of $\mathbb{D}$: why does $\mathcal{A}$ survive? Take $Y \in L[G]$. Let $\dot{Y}$ be a name for Y.
  - ~~First attempt: let α be such that $\dot{Y} \in M_\alpha$. Then $Y \in M_\alpha[G]$, so apply condition (4′) and we are done.~~
  - Second attempt: Let N be a countable model such that $\dot{Y} \in N$.

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    - Second attempt: Let N be a countable model such that $\dot{Y} \in N$. Now let α be such that $M_\alpha \cap \omega^\omega = N \cap \omega^\omega$. Then N and M_α contain the same P_ξ for $\xi \leq \alpha$ and agree on splitting reals. Hence (4′) applies also with N instead of M_α. □

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3. Does $\mathfrak{h} > \aleph_1$ imply "$\nexists \mathbf{\Sigma}^1_2$ MAD"? (Raghavan's conjecture.)

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4. Related conjecture: does "all $\mathbf{\Sigma}^1_2$ sets are Ramsey" imply "$\nexists \mathbf{\Sigma}^1_2$ MAD"?

Thank you!

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