MAD families and the projective hierarchy

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MAD families

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2. A family $\mathcal{A} \subseteq [\omega]^\omega$ is called \textit{almost disjoint} (a.d.) if

$$\forall A, B \in \mathcal{A} \ (A \text{ and } B \text{ are a.d.})$$
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3. A family $\mathcal{A} \subseteq [\omega]^{\omega}$ is called maximal almost disjoint (MAD) if it is an infinite a.d. family and maximal with this property.
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3. A family $\mathcal{A} \subseteq [\omega]^\omega$ is called *maximal almost disjoint* (MAD) if it is an infinite a.d. family and maximal with this property.

MAD families can be constructed in ZFC using a well-ordering of $[\omega]^\omega$. 
Via $[\omega]^\omega \cong 2^\omega$, we can talk about the complexity MAD families (descriptive set theory).
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**Fact**

*If \(V = L\) then there is a \(\Sigma^1_2\) MAD family (use the \(\Sigma^1_2\) well-ordering of \([\omega]^\omega\)).*
Complexity

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**Theorem (Mathias 1977)**

*There is no analytic MAD family.*

**Fact**

*If $V = L$ then there is a $\Sigma^1_2$ MAD family (use the $\Sigma^1_2$ well-ordering of $[\omega]^\omega$).*

**Theorem (Miller 1989)**

*If $V = L$ then there is a $\Pi^1_1$ MAD family.*
Question: are there $\Pi^1_1/\Sigma^1_2$ MAD families in models larger than $L$?
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Definition

Let $\mathbb{P}$ be a (proper) forcing. A MAD family $\mathcal{A}$ is called $\mathbb{P}$-indestructible if it remains a MAD family in $V^\mathbb{P}$. 
Preservation of $\Sigma^1_2$ definition

Fact

If $A \in L$ is a $\Sigma^1_2$ $\mathbb{P}$-indestructible MAD family, then in $L^\mathbb{P}$ it is still $\Sigma^1_2$. 
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If $\mathcal{A} \in L$ is a $\Sigma^1_2$ $\mathbb{P}$-indestructible MAD family, then in $L^\mathbb{P}$ it is still $\Sigma^1_2$.

Proof.

If $\phi(x)$ is $\Sigma^1_2$ and defines $\mathcal{A}$ in $L$, then "$\phi(x) \land x \in L$" is also $\Sigma^1_2$ and defines $\mathcal{A}$ in $L^\mathbb{P}$.
Preservation of $\Pi_1^1$ definition

Fact (Friedman & Zdomskyy 2010)

If $A \in L$ is a $\Pi_1^1$ $P$-indestructible MAD family, then in $L^P$ it is still $\Pi_1^1$. 

Proof. Let $\varphi(x)$ define $A$ in $L$, then in $L^P$ it defines a larger family $A'$. But the statement "$\forall x \forall y (\varphi(x) \land \varphi(y) \rightarrow x \cap y \text{ is finite})" has complexity $\Pi_1^2$ and holds in $L$, so by Shoenfield absoluteness, it holds in $L^P$. Therefore $A'$ is a.d., but since $A \subseteq A'$ and $A$ is maximal, it must be the case that $A = A'$. Therefore $A$ has a $\Pi_1^1$ definition.
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Well-known: for (iterations of) many standard forcing notions $\mathbb{P}$, including Cohen-, random-, Sacks- and Miller forcing, there are $\mathbb{P}$-indestructible MAD families.
Models of $\neg\text{CH}$

- **Well-known:** for (iterations of) many standard forcing notions $\mathbb{P}$, including Cohen-, random-, Sacks- and Miller forcing, there are $\mathbb{P}$-indestructible MAD families.
- **Also known:** such constructions can be made $\Pi^1_1$ in $L$ (Miller 1989, Kastermans et al 2008).
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**Corollary**

\[
\text{CON}(\neg \text{CH} + \exists \Pi^1_1 \text{ MAD}).
\]
Dominating reals

- **Well-known:** if $\mathcal{A}$ is MAD and $\mathbb{P}$ adds a dominating real (i.e., a real which dominates all ground model reals), then $\mathcal{A}$ is no longer MAD in $V^\mathbb{P}$.
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Questions:

- Is “$b > \aleph_1 + \exists \Sigma^1_2$ MAD” consistent?
- Is “$b > \aleph_1 + \exists \Pi^1_1$ MAD” consistent?
Friedman & Zdomskyy 2010: \( \text{CON}(\mathfrak{b} > \aleph_1 + \exists \Pi^1_2 \text{ MAD family}) \).
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- Raghavan (unpublished): if $t > \aleph_1$ then $\not\exists \Sigma^1_2 \text{ MAD}$. 
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To build a model of "$b > \aleph_1 + \exists \Sigma^1_2 \text{ MAD}$", previous methods don’t suffice, because they only produce MAD families $\mathcal{A} \subseteq L$. 
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To build a model of “$\mathfrak{b} > \aleph_1 + \exists \Sigma^1_2 \text{ MAD}$”, previous methods don’t suffice, because they only produce MAD families $\mathcal{A} \subseteq L$.

To avoid this problem, we consider MAD families defined by $\aleph_1$-unions of perfect sets.
An $\aleph_1$-perfect MAD is a MAD family $A$ such that $A = \bigcup \{P_\alpha \mid \alpha < \aleph_1\}$, where $P_\alpha$ is a perfect set.
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An $\aleph_1$-perfect MAD $\mathcal{A}$ is $\mathbb{P}$-indestructible if in $V^\mathbb{P}$, $\mathcal{A}^{V^\mathbb{P}} := \bigcup \{ P^{V^\mathbb{P}}_\alpha \mid \alpha < \aleph_1 \}$ is MAD.
\(\aleph_1\)-perfect MAD

**Definition**

1. An \(\aleph_1\)-perfect MAD is a MAD family \(A\) such that 
   \[A = \bigcup \{ P_\alpha \mid \alpha < \aleph_1 \},\]
   where \(P_\alpha\) is a perfect set.

2. An \(\aleph_1\)-perfect MAD \(A\) is \(P\)-indestructible if in \(V^P\),
   \[A^V := \bigcup \{ P^V_\alpha \mid \alpha < \aleph_1 \}\]
   is MAD.

NB. If \(P\) adds a dominating real, it will destroy the \(\aleph_1\)-union of the *old* perfect sets, but not necessarily that of the *new* perfect sets.
Main result

Theorem (Brendle-Kh.)

\[ \text{CON}(b > \aleph_1 + \exists \Sigma^1_2 \text{ MAD}). \]
Main result

**Theorem (Brendle-Kh.)**

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Recall:

- Hechler forcing $\mathbb{D}$ consist of conditions $(s, f) \in \omega^{<\omega} \times \omega^\omega$ with $s \subseteq f$, ordered by

$$(s', f') \leq (s, f) \iff s \subseteq s' \text{ and } f \leq f'$$
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- \( \mathbb{D} \) generically adds a dominating real.
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  \[(s', f') \leq (s, f) \iff s \subseteq s' \text{ and } f \leq f' \]
- \( \mathbb{D} \) generically adds a dominating real.
- \( \mathbb{D} \) preserves splitting families: if \( S \subseteq [\omega]^\omega \) is a splitting family in \( V \) then it is still a splitting family in \( V^{\mathbb{D}} \).
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Theorem (Brendle-Kh.)

CON($\mathfrak{b} > \aleph_1 + \exists \Sigma^1_2$ MAD).

Recall:
- Hechler forcing $\mathbb{D}$ consist of conditions $(s, f) \in \omega^{<\omega} \times \omega^\omega$ with $s \subseteq f$, ordered by
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- $\mathbb{D}$ generically adds a dominating real.
- $\mathbb{D}$ preserves splitting families: if $S \subseteq [\omega]^{\omega}$ is a splitting family in $V$ then it is still a splitting family in $V^{\mathbb{D}}$.

To prove the theorem, it suffices to construct a $\mathbb{D}$-indestructible, $\Sigma^1_2$-definable, $\aleph_1$-perfect MAD family in $L$. 
Proof

Proof: By simultaneous induction, construct a sequence $\langle P_\alpha \mid \alpha < \omega_1 \rangle$ of perfect a.d. sets, and an increasing sequence $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ of countable models, such that the following conditions are satisfied:

1. $\bigcup_{\alpha < \omega_1} M_\alpha$ covers all the reals,
2. $P_\alpha \in M_\alpha$ for all $\alpha$,
3. For all $\alpha, \beta$ and all $a \in P_\alpha$, $b \in P_\beta$, $|a \cap b| < \omega$, and
4. For all $\alpha$ and all $Y \in \left[ \omega \right]_\omega \cap M_\alpha$: if $Y$ is a.d. from $\bigcup_{\xi \leq \alpha} P_\xi$, then it is not a.d. from $P_{\alpha+1}$.

To satisfy (4), $P_{\alpha+1}$ is constructed from Cohen reals over $M_\alpha$ and using the fact that Cohen reals are splitting reals.

(4') Let $G$ be generic for the $\aleph_2$-iteration of $D$. For all $\alpha$ and all $Y \in \left[ \omega \right]_\omega \cap M_\alpha [G]$: if $Y$ is a.d. from $\bigcup_{\xi \leq \alpha} P_\xi [G]$, then it is not a.d. from $P_{\alpha+1} [G]$. 
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(4) Let $G$ be generic for the $\aleph_2$-iteration of $\mathcal{D}$. For all $\alpha$ and all $Y \in \mathcal{P}(\omega) \cap M_\alpha \setminus G$: if $Y$ is a.d. from $\bigcup_{\xi \leq \alpha} P_\xi$ $\mathcal{P}(\mathcal{V} \setminus G)_\xi$, then it is not a.d. from $P_{\alpha+1}$. 
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4. For all $\alpha$ and all $Y \in [\omega]^{\omega} \cap M_\alpha$: if $Y$ is a.d. from $\bigcup_{\xi \leq \alpha} P_\xi$, then it is not a.d. from $P_{\alpha+1}$.

(4') Let $G$ be generic for the $\aleph_2$-iteration of $D$. For all $\alpha$ and all $Y \in [\omega]^{\omega} \cap M_\alpha$: if $Y$ is a.d. from $\bigcup_{\xi \leq \alpha} P_{V[G]} \xi$, then it is not a.d. from $P_{V[G]}_{\alpha+1}$.
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4. For all \( \alpha \) and all \( Y \in [\omega]^\omega \cap M_\alpha \): if \( Y \) is a.d. from \( \bigcup_{\xi \leq \alpha} P_\xi \), then it is not a.d. from \( P_{\alpha+1} \).

To satisfy (4), \( P_{\alpha+1} \) is constructed from Cohen reals over \( M_\alpha \) and using the fact that Cohen reals are splitting reals.
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To satisfy (4), $P_{\alpha+1}$ is constructed from Cohen reals over $M_\alpha$ and using the fact that Cohen reals are *splitting reals*.

(4') Let $G$ be generic for the $\aleph_2$-iteration of $\mathbb{D}$.
For all $\alpha$ and all $Y \in [\omega]^{\omega} \cap M_\alpha[G]$: if $Y$ is a.d. from $\bigcup_{\xi \leq \alpha} P_\xi^{V[G]}$, then it is *not* a.d. from $P_{\alpha+1}^{V[G]}$. 

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Proof

Let $\mathcal{A} := \bigcup_{\alpha < \omega_1} P_\alpha$. Note that $\mathcal{A}$ can easily be made $\Sigma^1_2$. 
Proof

Let \( \mathcal{A} := \bigcup_{\alpha < \omega_1} P_\alpha \). Note that \( \mathcal{A} \) can easily be made \( \Sigma^1_2 \).

- In L: why is \( \mathcal{A} \) MAD?

First attempt: let \( \alpha \) be such that \( \dot{Y} \in M_\alpha \). Then \( Y \in M_\alpha[G] \), so apply condition (4') and we are done.

Problem: \( \bigcup_{\alpha < \omega_1} M_\alpha \) cannot cover all names for reals, because we are dealing with an \( \aleph_2 \)-iteration.

First attempt: let \( \alpha \) be such that \( \dot{Y} \in M_\alpha \). Then \( Y \in M_\alpha[G] \), so apply condition (4') and we are done.

Second attempt: Let \( N \) be a countable model such that \( \dot{Y} \in N \). Now let \( \alpha \) be such that \( M_\alpha \cap \omega = N \cap \omega \). Then \( N \) and \( M_\alpha \) contain the same \( P_\xi \) for \( \xi \leq \alpha \) and agree on splitting reals. Hence (4') applies also with \( N \) instead of \( M_\alpha \).
Proof

Let $\mathcal{A} := \bigcup_{\alpha < \omega_1} P_\alpha$. Note that $\mathcal{A}$ can easily be made $\Sigma^1_2$.

- In L: why is $\mathcal{A}$ MAD? Take $Y \in [\omega]^\omega$. Let $\alpha$ be such that $Y \in M_\alpha$, apply condition (4) and we are done.
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- In L: why is $\mathcal{A}$ MAD? Take $Y \in [\omega]^\omega$. Let $\alpha$ be such that $Y \in M_\alpha$, apply condition (4) and we are done.
- In the $\aleph_2$-iteration of $D$: why does $\mathcal{A}$ survive?
Proof

Let $\mathcal{A} := \bigcup_{\alpha < \omega_1} P_\alpha$. Note that $\mathcal{A}$ can easily be made $\Sigma^1_2$.

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- In the $\aleph_2$-iteration of $\mathbb{D}$: why does $\mathcal{A}$ survive? Take $Y \in L[G]$. Let $\dot{Y}$ be a name for $Y$. 

First attempt: let $\alpha$ be such that $\dot{Y} \in M_\alpha$. Then $Y \in M_\alpha[G]$, so apply condition (4) and we are done.

Problem: $\bigcup_{\alpha < \omega_1} M_\alpha$ cannot cover all names for reals, because we are dealing with an $\aleph_2$-iteration.

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- In the $\aleph_2$-iteration of $D$: why does $A$ survive? Take $Y \in L[G]$. Let $\dot{Y}$ be a name for $Y$.

  - First attempt: let $\alpha$ be such that $\dot{Y} \in M_\alpha$. Then $Y \in M_\alpha[G]$, so apply condition (4') and we are done.
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Let $\mathcal{A} := \bigcup_{\alpha < \omega_1} P_\alpha$. Note that $\mathcal{A}$ can easily be made $\Sigma^1_2$.

- In L: why is $\mathcal{A}$ MAD? Take $Y \in [\omega]^{\omega}$. Let $\alpha$ be such that $Y \in M_\alpha$, apply condition (4) and we are done.
- In the $\aleph_2$-iteration of $\mathbb{D}$: why does $\mathcal{A}$ survive? Take $Y \in L[G]$. Let $\dot{Y}$ be a name for $Y$.
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  - Second attempt: Let $N$ be a countable model such that $\dot{Y} \in N$. Then $N$ and $M_\alpha$ contain the same $P_\xi$ for $\xi \leq \alpha$ and agree on splitting reals. Hence (4') applies also with $N$ instead of $M_\alpha$. 

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Proof

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- In $L$: why is $\mathcal{A}$ MAD? Take $Y \in [\omega]^{\omega}$. Let $\alpha$ be such that $Y \in M_{\alpha}$, apply condition (4) and we are done.

- In the $\aleph_2$-iteration of $\mathbb{D}$: why does $\mathcal{A}$ survive? Take $Y \in L[G]$. Let $\dot{Y}$ be a name for $Y$.
  - First attempt: let $\alpha$ be such that $\dot{Y} \in M_{\alpha}$. Then $Y \in M_{\alpha}[G]$, so apply condition $(4')$ and we are done.
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Proof

Let \( A := \bigcup_{\alpha < \omega_1} P_\alpha \). Note that \( A \) can easily be made \( \Sigma^1_2 \).

- In \( L \): why is \( A \) MAD? Take \( Y \in [\omega]^\omega \). Let \( \alpha \) be such that \( Y \in M_\alpha \), apply condition (4) and we are done.

- In the \( \aleph_2 \)-iteration of \( D \): why does \( A \) survive? Take \( Y \in L[G] \). Let \( \dot{Y} \) be a name for \( Y \).
  - First attempt: let \( \alpha \) be such that \( \dot{Y} \in M_\alpha \). Then \( Y \in M_\alpha[G] \), so apply condition (4′) and we are done.
  - Second attempt: Let \( N \) be a countable model such that \( \dot{Y} \in N \). Now let \( \alpha \) be such that \( M_\alpha \cap \omega^\omega = N \cap \omega^\omega \). Then \( N \) and \( M_\alpha \) contain the same \( P_\xi \) for \( \xi \leq \alpha \) and agree on splitting reals. Hence (4′) applies also with \( N \) instead of \( M_\alpha \).

\( \square \)
Questions

Open Questions:

1. Is \( b > \aleph_1 + \exists \Pi^1_1 \text{ MAD} \) consistent?

2. More general: is \( \exists \Pi^1_1 \text{ MAD} \) equivalent to \( \exists \Sigma^1_2 \text{ MAD} \)?

3. Does \( h > \aleph_1 \) imply \( \exists \Sigma^1_2 \text{ MAD} \)? (Raghavan’s conjecture.)

4. Related conjecture: does “all \( \Sigma^1_2 \) sets are Ramsey” imply \( \exists \Sigma^1_2 \text{ MAD} \)?
Open Questions:

1. Is “\( b > \aleph_1 + \exists \Pi^1_1 \) MAD” consistent?

   Problem: if we write \( \mathcal{A} = \bigcup \{ P_x \mid x \in I \} \), where \( I \) is a family of reals coding the perfect sets, then \( I \) can be made \( \Pi^1_1 \) using the methods of (Miller 1989). But that implies only that \( \mathcal{A} \) is \( \Sigma^1_2 \).
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2. More general: is “$\exists\Pi^1_1$ MAD” equivalent to “$\exists\Sigma^1_2$ MAD”?

3. Does $h > \aleph_1$ imply “$\nexists\Sigma^1_2$ MAD”? (Raghavan’s conjecture.)

4. Related conjecture: does “all $\Sigma^1_2$ sets are Ramsey” imply “$\nexists\Sigma^1_2$ MAD”? 
Open Questions:

1. Is \( b > \aleph_1 + \exists \Pi^1_1 \text{ MAD} \) consistent?

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2. More general: is \( \exists \Pi^1_1 \text{ MAD} \) equivalent to \( \exists \Sigma^1_2 \text{ MAD} \)?

3. Does \( \aleph > \aleph_1 \) imply \( \# \Sigma^1_2 \text{ MAD} \)? (Raghavan’s conjecture.)
Open Questions:

1. Is \( \mathfrak{b} > \aleph_1 + \exists \Pi^1_1 \text{ MAD} \) consistent?

   Problem: if we write \( \mathcal{A} = \bigcup \{ P_x \mid x \in I \} \), where \( I \) is a family of reals coding the perfect sets, then \( I \) can be made \( \Pi^1_1 \) using the methods of (Miller 1989). But that implies only that \( \mathcal{A} \) is \( \Sigma^1_2 \).

2. More general: is \( \exists \Pi^1_1 \text{ MAD} \) equivalent to \( \exists \Sigma^1_2 \text{ MAD} \)?

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4. Related conjecture: does “all \( \Sigma^1_2 \) sets are Ramsey” imply \( \# \Sigma^1_2 \text{ MAD} \)?
Thank you!

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