# Do infinitely often equal trees add Cohen reals?

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# Infinitely often equal reals

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- $A \subseteq \omega^\omega$ is a **countably infinitely often equal (countably ioe) family** iff

$$\forall \{x_i \mid i < \omega\}\, \exists y \in A\, (y \text{ is ioe to every } x_i).$$

# Infinitely often equal $\sigma$-ideal

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  The generators are $K_{x,n} := \{y \in \omega^\omega \mid \forall m \geq n\, (x(m) \neq y(m))\}$, for $x \in \omega^\omega$ and $n \in \omega$.

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- $\mathcal{B}(\omega^\omega)/\mathfrak{I}_{\mathrm{ioe}}$ as a **notion of forcing** is proper, preserves category (every non-meager set remains non-meager in the generic extension) and has the continuous reading of names, by a result of Zapletal.

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- $\mathcal{B}(\omega^\omega)/\mathfrak{I}_{\mathrm{ioe}}$ generically adds an **ioe real** (i.e., a real which is ioe to all ground model reals). That is because it avoids all ground-model-coded Borel $\mathfrak{I}_{\mathrm{ioe}}$-small sets.

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**Theorem (Spinas 2008)**

For every analytic $A$, either $A \in \mathfrak{I}_{\mathrm{ioe}}$ or $A$ contains $[T]$ for some full-splitting Miller tree $T$.

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*So we decided to look at Spinas's theorem again.*

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# Infinitely often equal trees

Let $\mathbb{IE}$ denote the partial order of ioe-trees. Then

$$\mathbb{IE} \hookrightarrow_d \mathcal{B}(\omega^\omega)/\mathfrak{I}_{\text{ioe}}$$

(in fact we have two alternative proofs of this theorem, one using a Cantor-Bendixson argument and one using games).

**Thursday @ 17:30** *Giorgio Laguzzi will give a talk on some of our main results, such as the strength of this perfect-set-theorem, and some regularity properties related to* $\mathbb{IE}$ *as well as the mistaken full-splitting Miller tree forcing.*

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**Question**

Does $\mathbb{IE}$ add Cohen reals?

# Half a Cohen real

**Theorem (Bartoszyński)**

*If $x$ is ioe over $M$ and $y$ is ioe over $M[x]$ then in $M[x][y]$ there exists a Cohen real over $M$.*

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**Question (Fremlin)**

Is there a forcing adding an ioe real without adding a Cohen real?

# Zapletal's soluton

**Theorem (Zapletal 2013)**

*Let $X$ be a compact metrizable space which is infinite-dimensional, and all of its compact subsets are either infinite-dimensional or zero-dimensional. Let $\mathfrak{I}$ be the $\sigma$-ideal $\sigma$-generated by the compact zero-dimensional subsets of $X$. Then $\mathcal{B}(X)/\mathfrak{I}$ adds an ioe real but not a Cohen real.*

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**If** we can prove "for every $S \in \mathbb{IE}$, every continuous $f : [S] \to \omega^\omega$ there is $T \leq S$ such that $f``[T]$ is meager", then $\mathbb{IE}$ does not add Cohen reals.

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**Why?** Using **continuous reading of names**, for every name for a real $\dot{x}$ there is $S \in \mathbb{IE}$ and continuous $f : [S] \to \omega^\omega$ such that $S \Vdash \dot{x} = f(\dot{x}_G)$. If $T \leq S$ is such that $f``[T] \in \mathcal{M}$ then $T \Vdash$ "$\dot{x} \in f``[T] \in \mathcal{M}$" and hence $T \Vdash$ "$\dot{x}$ is not Cohen".

# Meager image

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The proof of this theorem is rather weird!

**Lemma**

*If* $\operatorname{add}(\mathcal{M}) < \operatorname{cov}(\mathcal{M})$ *then* $\mathbb{IE}$ *has the meager image property.*

**Corollary**

$\mathbb{IE}$ *has the meager image property.*

# Proofs

## Proof of Lemma $\Rightarrow$ Corollary

What is the complexity of "$\forall f : \omega^\omega \to \omega^\omega$ continuous $\exists T \in \mathbb{IE}$ such that $f``[T] \in \mathcal{M}$."?

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- "$f : \omega^\omega \to \omega^\omega$ is a continuous function" can be expressed as "$f' : \omega^{<\omega} \to \omega^{<\omega}$ is continuous (monotone and unbounded along each real)", which is $\mathbf{\Pi}^1_1$ on the real $f'$.
- "$T \in \mathbb{IE}$" is arithmetic on the code of $T$.
- $f``[T]$ is an analytic set whose code is recursive in $f'$ and $T$.
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- "$T \in \mathbb{IE}$" is arithmetic on the code of $T$.
- $f``[T]$ is an analytic set whose code is recursive in $f'$ and $T$.
- For an analytic set to be meager is $\mathbf{\Pi}^1_1$.

So the statement "$\mathbb{IE}$ has the meager image property" is $\mathbf{\Pi}^1_3$. Go to any forcing extension satisfying $\mathrm{add}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$ (e.g add $\omega_2$ Cohen reals), apply the Lemma and conclude that $\mathbb{IE}$ has the meager image property in the ground model by **downward $\mathbf{\Pi}^1_3$-absoluteness**. $\square$

# Proofs

**Lemma**

*If* $\mathrm{add}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$ *then* $\mathbb{IE}$ *has the meager image property.*

**Idea of proof:**

- Let $\mathrm{add}(\mathfrak{I}_{\mathrm{ioe}}, \mathbb{IE})$ be the least size of a family $\{X_\alpha \mid \alpha < \kappa\}$ such that $X_\alpha \in \mathfrak{I}_{\mathrm{ioe}}$ but there is no $\mathbb{IE}$-tree $T$ completely contained in the **complement** of $\bigcup_{\alpha<\kappa} X_\alpha$.
- Prove that $\mathrm{cov}(\mathcal{M}) \leq \mathrm{add}(\mathfrak{I}_{\mathrm{ioe}}, \mathbb{IE})$.
- Assume $\mathbb{IE}$ does **not** have the meager image property: then there is $f : \omega^\omega \to \omega^\omega$ such that $f``[T]$ is not meager for all $T \in \mathbb{IE}$. This is equivalent to saying that $f$**-preimages of meager sets are** $\mathfrak{I}_{\mathrm{ioe}}$**-small**. From this it (essentially) follows that $\mathrm{add}(\mathfrak{I}_{\mathrm{ioe}}, \mathbb{IE}) \leq \mathrm{add}(\mathcal{M})$.
- This contradicts $\mathrm{add}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$. □

# Homogeneity

**Recall:** we would like to prove:

"for every $S \in \mathbb{IE}$, every continuous $f : [S] \to \omega^\omega$
there is $T \leq S$ such that $f``[T]$ is meager".

**In other words:**

"$\mathbb{IE}$ has the meager image property below $S$, for every $S \in \mathbb{IE}$".

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It is sufficient for $\mathfrak{I}_{\mathrm{ioe}}$ to be **homogeneous**.

# Goldstern-Shelah tree

**Lemma (Goldstern-Shelah 1994)**

*There exists* $T^{GS} \in \mathbb{IE}$ *such that every* $T \leq T^{GS}$ *is an* **almost-full-splitting Miller tree**, *i.e.,* $\forall t \in T\, \exists s \supseteq t\, (\forall^\infty n\, (s^\frown \langle n \rangle \in T))$.

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**Consequences:**

1. $T^{GS} \Vdash$ "there is a Cohen real".
2. $\mathfrak{I}_{\mathrm{ioe}}$ is **not** homogeneous.
3. "For every $S$, $\mathbb{IE}$ has the meager image property below $S$" is **false.**

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3. "For every $S$, $\mathbb{IE}$ has the meager image property below $S$" is **false.**

However, what could be true is "There exists $T_0$ s.t. for every $S \leq T_0$, $\mathbb{IE}$ has the meager image property below $S$." Then $T_0 \Vdash$ "there are no Cohen reals".

# Conclusion

- If we can find $T_0$ such that for every $S \leq T_0$ $\mathfrak{I}_{\mathrm{ioe}} \restriction S$ is homogeneous, we are done.
- On the other hand, if trees like $T^{GS}$ are dense in $\mathbb{IE}$, we are in trouble.

**Question**

Is there $T_0 \in \mathbb{IE}$ forcing that no Cohen reals are added?

## Thank you!

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