Regularity properties on the generalized reals

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joint with Sy Friedman and Vadim Kulikov

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- All analytic sets satisfy the Baire property (Suslin 1917).
- "All projective sets satisfy the Baire property" is independent of ZFC (Gödel 1938 + Solovay 1970).

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Call a subset κ -meager iff it is the κ -union of nowhere dense sets. Say that A has the κ -Baire property iff it is equal to an **open** set modulo a κ -meager set.

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Generalize **descriptive set theory** in the standard way:

- Borel = smallest collection containing open sets and closed under complements and κ -unions.
- Σ_1^1 = projections of closed.
- $\Pi_n^1 = \text{complements of } \Sigma_n^1$.

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Baire property for generalized projective sets

Observation

All (κ -)Borel sets have the κ -Baire property.

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Idea: let C denote the club filter on κ , considered as a subset of 2^{κ} , i.e.,

$$C = \{x \in 2^{\kappa} \mid \{i < \kappa \mid x(i) = 1\} \text{ contains a club}\}.$$

Note that:

- "To be closed" is (topologically) closed.
- "To be unbounded" is G_{δ} .
- \Rightarrow "To be in the club filter" is Σ_1^1 .

Show that C does not have the κ -Baire property (we will see a more general proof later).

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Theorem (Friedman-Hyttinen-Kulikov 2014)

A κ^+ -product of κ -Cohen forcing (forcing with $2^{<\kappa}$) with supports of size $<\kappa$, forces that all Δ_1^1 sets have the κ -Baire property.

(Remember that $\Delta_1^1 \neq \text{Borel}$).

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(Remember that $\Delta_1^1 \neq \text{Borel}$).

Also, it is easy to see that in L there is a Δ_1^1 set without the κ -Baire property.

So $\Delta_1^1(\kappa$ -Baire) is independent.

In the classical setting, people have studied many regularity properties: Lebesgue measure, Ramsey property, Sacks property etc. A lot of them can be cast in a unifying framework in terms of **forcing partial orders** (Brendle, Löwe, Ikegami, Kh, Laguzzi).

There is a rich theory of such properties for projective sets beyond the analytic (Δ_2^1 , Σ_2^1 etc.)

We wanted to conduct a systematic study of what happens with such properties in the setting of **generalized reals**.

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Why is this interesting?

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Some possible answers...

- Applications to forcing theory.
- Understanding "what makes ω so special".
- Importance of the club-filter.
- Understanding the importance of "absoluteness" in DST.
- Developing new forcing techniques

• ...

Classical vs. Generalized DST

Classical DST	Generalized DST
$Borel = \mathbf{\Delta}_1^1.$	$Borel \neq \mathbf{\Delta}_1^1.$
Σ_1^1 -absoluteness for all models and Shoenfield absoluteness for models containing ω_1 .	Σ_1^1 -absoluteness may fail even for forcing extensions (destroy station- ary set by shooting club); however, it holds for $<\kappa$ -closed forcing.
Σ_2^1 -good w.o. of the reals in <i>L</i> .	Σ_1^1 -good w.o. of the generalized reals in L .
"Proper forcing" is well-understood.	" κ -proper forcing" is not well- understood and no general iteration theorems.

Definition

We call a forcing poset $\mathbb{P} \kappa$ -tree-like if the conditions are trees on κ^{κ} or 2^{κ} , ordered by inclusion, with some additional assumptions:

- **1** If $T \in \mathbb{P}$ and $\sigma \in T$ then $T \uparrow \sigma \in \mathbb{P}$.
- All *T* ∈ ℙ are pruned (no terminal nodes) and <κ-closed (increasing sequences of length < κ of nodes in *T* have a limit in *T*).
- **3** The definition of \mathbb{P} is absolute.



• κ -Cohen \mathbb{C}_{κ} : basic open sets $[\sigma]$ for $\sigma \in \kappa^{<\kappa}$ or $2^{<\kappa}$.

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- κ -Cohen \mathbb{C}_{κ} : basic open sets $[\sigma]$ for $\sigma \in \kappa^{<\kappa}$ or $2^{<\kappa}$.
- κ -Sacks \mathbb{S}_{κ} : trees $T \subseteq 2^{<\kappa}$ s.t.
 - every node has a splitting extension, and
 - if $\{\sigma_i \mid i < \lambda\}$ is an increasing sequence of splitting nodes of length $\lambda < \kappa$, then $\bigcup_{i < \lambda} \sigma_i$ is a splitting node.

(Kanamori 1980)

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- κ -Miller \mathbb{M}_{κ} : forcing conditions are trees $T \subseteq \kappa^{<\kappa}$ s.t.
 - every node has a (club-)splitting extension, and
 - if $\{\sigma_i \mid i < \lambda\}$ is an increasing sequence of club-splitting nodes of length $\lambda < \kappa$, then $\bigcup_{i < \lambda} \sigma_i$ is a club-node.

(Friedman & Zdomskyy 2010)

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Other (more artificial?) examples:

- κ -Laver \mathbb{L}_{κ} : every $\sigma \in \mathcal{T}$ extending the stem is club-splitting.
- κ -Mathias \mathbb{R}_{κ} : uniform version of \mathbb{L}_{κ} .
- κ -Silver \mathbb{V}_{κ} : uniform version of \mathbb{S}_{κ} (only makes sense for inaccessible κ).

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NB: **random forcing** is missing from the list—we don't know how to generalize random forcing to generalized Baire spaces (cf. Giorgio's talk tomorrow).

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For $A \subseteq \kappa^{\kappa}$ or 2^{κ} , we follow the abstract approach of Ikegami and define:

Definition

- *A* is \mathbb{P} -nowhere dense iff $\forall T \in \mathbb{P} \exists S \leq T ([S] \cap A = \emptyset)$.
- A is \mathbb{P} -meager iff it is the countable union of \mathbb{P} -null sets.
- A is P-measurable iff ∀T ∈ P∃S ≤ T ([S] ⊆* A or [S] ∩ A =* Ø), where ⊆* and =* stand for "modulo P-meager".

For $\mathbb{P} = \kappa$ -Cohen, this generalizes the Baire property.

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- Are Borel sets P-measurable?
- **2** Are Σ_1^1 -sets \mathbb{P} -measurable?
- **3** Are Δ_1^1 -sets \mathbb{P} -measurable?
- Imitating classical Δ_2^1 -theory on Δ_1^1 -level?

1. Are Borel sets P-measurable?

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1. Are Borel sets P-measurable?

In the ω^{ω} -setting we can use **forcing** and **Shoenfield absoluteness** to prove that all Σ_1^1 -sets are \mathbb{P} -measurable (for a wide class of \mathbb{P}). But in the generalized setting Shoenfield absoluteness may fail, so we need to rely on more primitive methods.

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Definition

 \mathbb{P} is **topological** iff $\{[T] \mid T \in \mathbb{P}\}$ forms a topology base on κ^{κ} (i.e., $T \perp S \Rightarrow [T] \cap [S] = \emptyset$).

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Topological or Axiom A

Definition

 \mathbb{P} satisfies Axiom A iff there are orderings $\{\leq_{\alpha} | \alpha < \kappa\}$, with $\leq_{0} = \leq$, satisfying:

• $T \leq_{\beta} S$ implies $T \leq_{\alpha} S$, for all $\alpha \leq \beta$.

2 If $\langle T_{\alpha} \mid \alpha < \lambda \rangle$ is a sequence of conditions, with $\lambda \leq \kappa$ (in particular $\lambda = \kappa$) satisfying $T_{\beta} \leq_{\alpha} T_{\alpha}$ for all $\alpha \leq \beta$, then there exists $T \in \mathbb{P}$ such that $T \leq_{\alpha} T_{\alpha}$ for all $\alpha < \lambda$.

So For all T ∈ P, D dense below T, and α < κ, there exists an E ⊆ D and S ≤_α T such that |E| ≤ κ and E is predense below S.

Definition

 \mathbb{P} satisfies **Axiom A**^{*} if in 3 of the definition above, additionally we have " $[S] \subseteq \bigcup \{[T] \mid T \in E\}$ ".

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If \mathbb{P} is topological then A is \mathbb{P} -measurable iff A has the Baire property in the \mathbb{P} -topology. In particular, Borel sets are \mathbb{P} -measurable.

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Lemma

If \mathbb{P} satisfies Axiom A^{*} then the algebra of \mathbb{P} -measurable sets is closed under κ -unions and -intersections. In particular, Borel sets are \mathbb{P} -measurable.

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If \mathbb{P} satisfies Axiom A^{*} then the algebra of \mathbb{P} -measurable sets is closed under κ -unions and -intersections. In particular, Borel sets are \mathbb{P} -measurable.

In all practical cases ${\ensuremath{\mathbb P}}$ satisfies one of the above conditions.

NB: This is completely analogous to the classical situation!

2. Are Σ_1^1 sets \mathbb{P} -measurable?

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Recall the club filter used by Halko & Shelah:

$$C = \{x \in 2^{\kappa} \mid \{i < \kappa \mid x(i) = 1\} \text{ contains a club}\}.$$

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Recall the **club filter** used by Halko & Shelah:

 $C = \{ x \in 2^{\kappa} \mid \{ i < \kappa \mid x(i) = 1 \} \text{ contains a club} \}.$

For $S \subseteq \kappa$ stationary, co-stationary, define:

 $C_{S} = \{ x \in \kappa^{\kappa} \mid \{ i < \kappa \mid x(i) \in S \} \text{ contains a club} \}.$

Clearly C_S is also Σ_1^1 .

Generalizing Halko-Shelah

Theorem (Friedman-Kh-Kulikov)

- **1** If \mathbb{P} is any tree-like forcing on 2^{κ} refining \mathbb{S}_{κ} , then C is not \mathbb{P} -measurable.
- 2 If \mathbb{P} is any tree-like forcing on κ^{κ} refining \mathbb{M}_{κ} , then C_{S} is not \mathbb{P} -measurable.

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Proof.

(1) Suppose *C* is \mathbb{P} -measurable, let $T \in \mathbb{P}$ be s.t. $[T] \subseteq^* C$ or $[T] \cap C =^* \emptyset$, w.l.o.g. the former. Let $\{X_i \mid i < \kappa\}$ be \mathbb{P} -nowhere dense sets such that $[T] \setminus C = \bigcup_{i < \kappa} X_i$. Construct a decreasing sequence of trees as follows:

- $T_0 := T$,
- $T_{i+1} \leq T_i$ is s.t. $[T_{i+1}] \cap X_i = \emptyset$ and $|\operatorname{stem}(T_{i+1})| > |\operatorname{stem}(T_i)|$,
- at limits λ , first let $T'_{\lambda} := \bigcap_{i < \lambda} T_i$, which is in \mathbb{P} by assumption. Choose $T_{\lambda} \leq T'_{\lambda}$ such that $\operatorname{stem}(T_{\lambda}) \supseteq \operatorname{stem}(T'_{\lambda}) \cap \langle 0 \rangle$.

Now $x := \bigcup_{i < \kappa} \operatorname{stem}(T_i)$ is a branch through $T, x \notin X_i$ for all $i < \kappa$, and x(i) = 0 for club-many $i < \kappa$, hence $x \notin C$ —contradiction.

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Proof.

(2) Proceed analogously, except that at limit stages choose $T_{\lambda} \leq T'_{\lambda}$ such that $\operatorname{stem}(T_{\lambda}) \supseteq \operatorname{stem}(T'_{\lambda}) \cap \langle \alpha \rangle$, where α is in S or not in S depending on what we want.

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Corollary

For all \mathbb{P} refining \mathbb{S}_{κ} or \mathbb{M}_{κ} , $\Sigma_1^1(\mathbb{P}$ -measurability) is false.

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3. Are Δ_1^1 sets \mathbb{P} -measurable?

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In L, use the Σ_1^1 -good wellorder to construct counterexamples to $\Delta_1^1(\mathbb{P})$, for any \mathbb{P} , by diagonalization.



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In L, use the Σ_1^1 -good wellorder to construct counterexamples to $\Delta_1^1(\mathbb{P})$, for any \mathbb{P} , by diagonalization.

Question: Is $\Delta_1^1(\mathbb{P}$ -measurability) consistent?

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Theorem (Friedman-Kh-Kulikov)

Let \mathbb{P} be a $<\kappa$ -closed, κ -tree-like forcing.

- Suppose P satisfies the κ⁺-c.c., and let P_{κ⁺} be the κ⁺-iteration of P with supports of size <κ. Then V^{P_{κ⁺}} ⊨ Δ¹₁(P-measurability).
- Suppose P satisfies Axiom A*, and let P_{κ+} be the κ⁺-iteration of P with supports of size ≤κ. Moreover, assume that for every x ∈ κ^κ ∩ V^{P_{κ+}}, there is α < κ⁺ such that x ∈ κ^κ ∩ V^{P_α}. Then V^{P_{κ+}} ⊨ Δ¹₁(P-measurability).

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All forcings we consider are $<\kappa$ -closed and satisfy either the κ^+ -c.c. or Axiom A*. However, the red condition is essentially about "preservation of κ -properness", which is a very difficult problem in the generalized setting.

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For the proof, we need a lemma which is proved similarly to the $\omega^\omega\text{-case}.$

Lemma

Let \mathbb{P} be as in the theorem. For every elementary submodel $M \prec \mathcal{H}_{\theta}$ of a sufficiently large \mathcal{H}_{θ} , with $|M| = \kappa$ and $M^{<\kappa} \subseteq M$, and for every $T \in \mathbb{P} \cap M$, there is $T' \leq T$ such that

$$[T'] \subseteq^* \{x \in \kappa^{\kappa} \mid x \text{ is } \mathbb{P}\text{-generic over } M\}.$$

(where \subseteq^* means "modulo \mathbb{P} -meager" and a κ -real x is \mathbb{P} -generic over M iff $\{S \in \mathbb{P} \cap M \mid x \in [S]\}$ is a \mathbb{P} -generic filter over M.)

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Proof of theorem

Proof.

In $V[\mathcal{G}_{\kappa^+}]$, let A be Δ_1^1 , defined by Σ_1^1 -formulas ϕ and ψ . Let $S \in \mathbb{P}$ be arbitrary. We must find $T \leq S$ such that $[T] \subseteq^* A$ or $T \cap A =^* \emptyset$.

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By assumption, there exists $\alpha < \kappa^+$ s.t. *S* and the parameters of ϕ and ψ belong to $V[G_{\alpha}]$. Moreover, there is a $\beta > \alpha$ s.t. *S* belongs to $G(\beta + 1)$. Let $x_{\beta+1}$ be the $(\beta + 1)$ -th generic real.

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In $V[G_{k+1}]$, either $\phi(x_{\beta+1})$ or $\psi(x_{\beta+1})$ holds. By symmetry, we may w.l.o.g. assume the former. Since (the iteration of) \mathbb{P} is $<\kappa$ -closed, we have Σ_1^1 -absoluteness between $V[G_{\kappa^+}]$ and $V[G_{\beta+1}]$. In particular, $V[G_{\beta+1}] \models \phi(x_{\beta+1})$. By the forcing theorem there exists $T \in V[G_{\beta}]$, $T \leq S$ and $T \Vdash_{\mathbb{P}} \phi(\dot{x}_{\text{gen}})$.

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In $V[G_{\kappa^+}]$, either $\phi(x_{\beta+1})$ or $\psi(x_{\beta+1})$ holds. By symmetry, we may w.l.o.g. assume the former. Since (the iteration of) \mathbb{P} is $<\kappa$ -closed, we have Σ_1^1 -absoluteness between $V[G_{\kappa^+}]$ and $V[G_{\beta+1}]$. In particular, $V[G_{\beta+1}] \models \phi(x_{\beta+1})$. By the forcing theorem there exists $T \in V[G_{\beta}], T \leq S$ and $T \Vdash_{\mathbb{P}} \phi(\dot{x}_{gen})$.

Take an elementary M of size κ containing T. By elementarity, $M \models "T \Vdash_{\mathbb{P}} \phi(\dot{x}_{gen})"$. Going back to $V[G_{\kappa^+}]$, use the previous lemma to find $T' \leq T$ such that $[T'] \subseteq^* \{x \mid x \text{ is } \mathbb{P}\text{-generic over } M\}$. Now note that if x is $\mathbb{P}\text{-generic over } M$ and $x \in [T]$, then $M[x] \models \phi(x)$. By upwards- Σ_1^1 -absoluteness between M and $V[G_{\kappa^+}]$ we conclude that $\phi(x)$ really holds. Since this was true for arbitrary $x \in [T']$, we obtain $[T'] \subseteq^* \{x \mid \phi(x)\} = A$.

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Corollary

Let \mathbb{P} be as in the assumption of the theorem. Then $\Delta_1^1(\mathbb{P}$ -measurability) is independent.

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Corollary

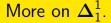
Let \mathbb{P} be as in the assumption of the theorem. Then $\Delta_1^1(\mathbb{P}$ -measurability) is independent.

The proof of the above theorem is related to classical proofs for Δ_2^1 sets. So a natural question is: how much of the theory for classical Δ_2^1 sets holds for Δ_1^1 sets in the generalized context?



4. Imitating classical Δ_2^1 -theory for Δ_1^1 -level?

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4. Imitating classical Δ_2^1 -theory for Δ_1^1 -level?

Theorem (Judah-Shelah 1989)

 Δ_2^1 (Baire property) holds iff for every $r \in \omega^{\omega}$ there exists a Cohen real over L[r].

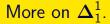


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Theorem (Judah-Shelah 1989)

 Δ_2^1 (Baire property) holds iff for every $r \in \omega^{\omega}$ there exists a Cohen real over L[r].

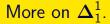
Does this hold for Δ_1^1 sets in the generalized context?



No!

Theorem (Friedman, Wu & Zdomskyy 2014)

Suppose κ is successor. There is a forcing iteration starting from L, in which cofinally many iterands have the κ^+ -c.c., such that in the extension the club filter is Δ_1^1 .

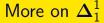


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One can verify that this iteration adds κ -Cohen reals cofinally often! Hence, in that model there are κ -Cohen reals over L[r], for every $r \in 2^{\kappa}$, however $\Delta_1^1(\kappa$ -Baire property) fails.

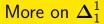


Still, there are a few things we can say.

Fact

 $\Delta_1^1(\kappa$ -Baire property) $\Rightarrow \forall r \in \kappa^{\kappa} \exists \kappa$ -Cohen real over L[r].

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$$\Delta^1_1(\kappa$$
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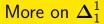
Lemma

Suppose κ inaccessible. Then $\Delta_1^1(\mathbb{M}_{\kappa}\text{-measurability}) \Rightarrow \forall r \in \kappa^{\kappa} \exists x (x \text{ is unbounded over } L[r]).$ $(This means <math>\{i \mid x(i) > y(i)\}$ is unbounded in κ , for every $y \in L[r]$).

Proof.

Based on the ω^{ω} -proof of Brendle & Löwe, but very technical.

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Lemma

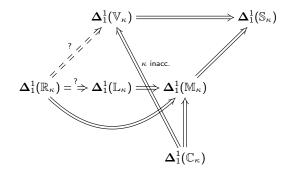
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Proof.

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We do not have any similar results for the other forcings notions.

Implication diagram on ${f \Delta}_1^1$ level



 $\mathbb{C}_{\kappa} = \mathsf{Cohen}, \, \mathbb{S}_{\kappa} = \mathsf{Sacks}, \, \mathbb{M}_{\kappa} = \mathsf{Miller}, \, \mathbb{L}_{\kappa} = \mathsf{Laver}, \, \mathbb{R}_{\kappa} = \mathsf{Mathias}, \, \mathbb{V}_{\kappa} = \mathsf{Silver}.$

The proofs are straightforward but quite technical.

Can we prove that some/any of these implications are strict, i.e., cannot be reversed?

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Theorem (Friedman-Kh-Kulikov)

Suppose κ is inaccessible. Then $\operatorname{Con}(\Delta_1^1(\mathbb{V}_{\kappa}\operatorname{-measurability}) + \neg \Delta_1^1(\mathbb{M}_{\kappa}))$.

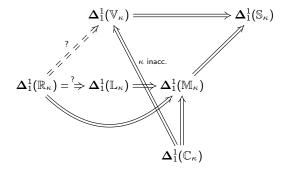
Proof.

Perform a κ^+ -iteration of κ -Silver forcing, starting in L, with supports of size κ . Then $\Delta^1_1(\mathbb{V}_{\kappa}$ -measurability) holds by our previous theorem. Next, show that " κ -properness" is preserved (similar to Kanamori's κ -Sacks). Using inaccessibility of κ , the iteration is " κ^{κ} -bounding". As a result, the generic extension does not satisfy the statement " $\forall r \exists x \ (x \text{ is unbounded over } \kappa^{\kappa} \cap L[r])$ ", so $\Delta^1_1(\mathbb{M}_{\kappa}$ -measurability) fails.

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Implication diagram on ${f \Delta}_1^1$ level

However, there are still many open questions!



 $\mathbb{C}_{\kappa}=\text{Cohen},\,\mathbb{S}_{\kappa}=\text{Sacks},\,\mathbb{M}_{\kappa}=\text{Miller},\,\mathbb{L}_{\kappa}=\text{Laver},\,\mathbb{R}_{\kappa}=\text{Mathias},\,\mathbb{V}_{\kappa}=\text{Silver}.$

Question

Are we looking at the right properties?

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All the properties we looked at are determined by **forcing** posets. We want these forcings to be $<\kappa$ -closed, so the trees $T \in \mathbb{P}$ are required to have a certain shape.

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Question

Are we looking at the right properties?

All the properties we looked at are determined by **forcing** posets. We want these forcings to be $<\kappa$ -closed, so the trees $T \in \mathbb{P}$ are required to have a certain shape.

In particular, all our trees T satisfy:

 $\forall x \in [T] (\{i < \kappa \mid x \mid i \text{ is a split-node of } T\} \text{ is club}).$

What if we drop this property?

Some results

• If we drop the assumption on κ -Sacks trees that "limits of split-nodes are split-nodes", we obtain a propery weaker than \mathbb{S}_{κ} -measurability, which consistently holds for all generalized projective sets, (Schlicht).

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- If we drop the assumption on κ-Silver trees that "splitting levels from a club" and replace it by "splitting levels form a stationary set", we obtain a property weaker than V_κ-measurability, which consistently holds for all generalized projective sets (Laguzzi).

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These weaker properties are not useful for forcing theory, because the corresponding forcing notions are **not** $<\kappa$ -closed.

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Work in progress (Friedman & Laguzzi)

Assume κ is measurable. Consider a version of Silver forcing in which the trees are required to split **on a set positive with respect to a normal measure on** κ . The corresponding forcing is κ -proper and $<\kappa$ -closed, and the corresponding regularity property is consistent for all projective sets.

Thank you!

Yurii Khomskii yurii@deds.nl

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