Regularity properties on the generalized reals

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The Baire property

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- All analytic sets satisfy the Baire property (Suslin 1917).
- *"All projective sets satisfy the Baire property"* is independent of ZFC (Gödel 1938 + Solovay 1970).

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Generalize **descriptive set theory** in the standard way:

- Borel = smallest collection containing open sets and closed under complements and κ-unions.
- $\mathbf{\Sigma}^1_1$ = projections of closed.
- $\mathbf{\Pi}^1_n$ = complements of $\mathbf{\Sigma}^1_n$.

Baire property for generalized projective sets

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All (κ-)Borel sets have the κ-Baire property.

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There is a $\mathbf{\Sigma}^1_1$ set without the κ-Baire property.

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Observation

All (κ-)Borel sets have the κ-Baire property.

Theorem (Halko-Shelah 2001)

There is a $\mathbf{\Sigma}^1_1$ set without the κ-Baire property.

Idea: let C denote the **club filter** on κ, considered as a subset of 2^κ, i.e.,

$$C = \{x \in 2^\kappa \mid \{i < \kappa \mid x(i) = 1\} \text{ contains a club}\}.$$

Note that:

- "To be closed" is (topologically) closed.
- "To be unbounded" is G_δ.
- $\Rightarrow$ "To be in the club filter" is $\mathbf{\Sigma}^1_1$.

Show that C does not have the κ-Baire property (we will see a more general proof later).

Baire property for generalized projective sets

Theorem (Friedman-Hyttinen-Kulikov 2014)

A κ^+-product of κ-Cohen forcing (forcing with $2^{<\kappa}$) with supports of size $<\kappa$, forces that all $\mathbf{\Delta}^1_1$ sets have the κ-Baire property.

(Remember that $\mathbf{\Delta}^1_1 \neq$ Borel).

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(Remember that $\mathbf{\Delta}^1_1 \neq$ Borel).

Also, it is easy to see that in L there is a $\mathbf{\Delta}^1_1$ set without the κ-Baire property.

So $\mathbf{\Delta}^1_1(\kappa\text{-Baire})$ is independent.

Combinatorial regularity properties

In the classical setting, people have studied many regularity properties: Lebesgue measure, Ramsey property, Sacks property etc. A lot of them can be cast in a unifying framework in terms of **forcing partial orders** (Brendle, Löwe, Ikegami, Kh, Laguzzi).

There is a rich theory of such properties for projective sets beyond the analytic ($\mathbf{\Delta}^1_2$, $\mathbf{\Sigma}^1_2$ etc.)

We wanted to conduct a systematic study of what happens with such properties in the setting of **generalized reals**.

Why is this interesting?

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Some possible answers...

- Applications to forcing theory.
- Understanding “what makes ω so special”.
- Importance of the club-filter.
- Understanding the importance of “absoluteness” in DST.
- Developing new forcing techniques
- ...

Classical vs. Generalized DST

Classical DST	Generalized DST
Borel $= \mathbf{\Delta}^1_1$.	Borel $\neq \mathbf{\Delta}^1_1$.
$\mathbf{\Sigma}^1_1$-absoluteness for **all** models and Shoenfield absoluteness for models containing ω_1.	$\mathbf{\Sigma}^1_1$-absoluteness may fail even for forcing extensions (destroy stationary set by shooting club); however, it holds for $<\kappa$-closed forcing.
$\mathbf{\Sigma}^1_2$-good w.o. of the reals in L.	$\mathbf{\Sigma}^1_1$-good w.o. of the generalized reals in L.
"Proper forcing" is well-understood.	"κ-proper forcing" is not well-understood and no general iteration theorems.

Definition

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We call a forcing poset $\mathbb{P}$ κ-**tree-like** if the conditions are trees on κ^κ or 2^κ, ordered by inclusion, with some additional assumptions:

1. If $T \in \mathbb{P}$ and $\sigma \in T$ then $T \uparrow \sigma \in \mathbb{P}$.
2. All $T \in \mathbb{P}$ are pruned (no terminal nodes) and $<\kappa$-closed (increasing sequences of length $< \kappa$ of nodes in T have a limit in T).
3. The definition of $\mathbb{P}$ is absolute.

Examples

- κ-Cohen $\mathbb{C}_\kappa$: basic open sets $[\sigma]$ for $\sigma \in \kappa^{<\kappa}$ or $2^{<\kappa}$.

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- κ-Sacks $\mathbb{S}_\kappa$: trees $T \subseteq 2^{<\kappa}$ s.t.
  - every node has a splitting extension, and
  - if $\{\sigma_i \mid i < \lambda\}$ is an increasing sequence of splitting nodes of length $\lambda < \kappa$, then $\bigcup_{i<\lambda} \sigma_i$ is a splitting node.

(Kanamori 1980)

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- κ-Miller $\mathbb{M}_\kappa$: forcing conditions are trees $T \subseteq \kappa^{<\kappa}$ s.t.
  - every node has a (club-)splitting extension, and
  - if $\{\sigma_i \mid i < \lambda\}$ is an increasing sequence of club-splitting nodes of length $\lambda < \kappa$, then $\bigcup_{i<\lambda} \sigma_i$ is a club-node.

  (Friedman & Zdomskyy 2010)

Other examples

Other (more artificial?) examples:

- κ-Laver $\mathbb{L}_\kappa$: every $\sigma \in T$ extending the stem is club-splitting.
- κ-Mathias $\mathbb{R}_\kappa$: uniform version of $\mathbb{L}_\kappa$.
- κ-Silver $\mathbb{V}_\kappa$: uniform version of $\mathbb{S}_\kappa$ (only makes sense for inaccessible κ).

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NB: **random forcing** is missing from the list—we don't know how to generalize random forcing to generalized Baire spaces (cf. Giorgio's talk tomorrow).

$\mathbb{P}$-measurability

For $A \subseteq \kappa^\kappa$ or 2^κ, we follow the abstract approach of Ikegami and define:

Definition

- A is $\mathbb{P}$-**nowhere dense** iff $\forall T \in \mathbb{P} \exists S \leq T\ ([S] \cap A = \varnothing)$.
- A is $\mathbb{P}$-**meager** iff it is the countable union of $\mathbb{P}$-null sets.
- A is $\mathbb{P}$-**measurable** iff $\forall T \in \mathbb{P} \exists S \leq T\ ([S] \subseteq^* A$ or $[S] \cap A =^* \varnothing)$, where $\subseteq^*$ and $=^*$ stand for "modulo $\mathbb{P}$-meager".

For $\mathbb{P} = \kappa$-Cohen, this generalizes the Baire property.

$\mathbb{P}$-measurability of projective sets

1. Are Borel sets $\mathbb{P}$-measurable?
2. Are $\mathbf{\Sigma}^1_1$-sets $\mathbb{P}$-measurable?
3. Are $\mathbf{\Delta}^1_1$-sets $\mathbb{P}$-measurable?
4. Imitating classical $\mathbf{\Delta}^1_2$-theory on $\mathbf{\Delta}^1_1$-level?

Borel

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In the ω^ω-setting we can use **forcing** and **Shoenfield absoluteness** to prove that all $\mathbf{\Sigma}^1_1$-sets are $\mathbb{P}$-measurable (for a wide class of $\mathbb{P}$). But in the generalized setting Shoenfield absoluteness may fail, so we need to rely on more primitive methods.

Topological or Axiom A

Definition

$\mathbb{P}$ is **topological** iff $\{[T] \mid T \in \mathbb{P}\}$ forms a topology base on κ^κ (i.e., $T \perp S \; \Rightarrow \; [T] \cap [S] = \varnothing$).

Topological or Axiom A

Definition

$\mathbb{P}$ satisfies **Axiom A** iff there are orderings $\{\leq_\alpha \mid \alpha < \kappa\}$, with $\leq_0 = \leq$, satisfying:

1. $T \leq_\beta S$ implies $T \leq_\alpha S$, for all $\alpha \leq \beta$.
2. If $\langle T_\alpha \mid \alpha < \lambda \rangle$ is a sequence of conditions, with $\lambda \leq \kappa$ (in particular $\lambda = \kappa$) satisfying $T_\beta \leq_\alpha T_\alpha$ for all $\alpha \leq \beta$, then there exists $T \in \mathbb{P}$ such that $T \leq_\alpha T_\alpha$ for all $\alpha < \lambda$.
3. For all $T \in \mathbb{P}$, D dense below T, and $\alpha < \kappa$, there exists an $E \subseteq D$ and $S \leq_\alpha T$ such that $|E| \leq \kappa$ and E is predense below S.

Definition

$\mathbb{P}$ satisfies **Axiom A*** if in 3 of the definition above, additionally we have "$[S] \subseteq \bigcup \{[T] \mid T \in E\}$".

Topological or Axiom A

Lemma

If $\mathbb{P}$ is topological then A is $\mathbb{P}$-measurable iff A has the Baire property in the $\mathbb{P}$-topology. In particular, Borel sets are $\mathbb{P}$-measurable.

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If $\mathbb{P}$ satisfies Axiom A^* then the algebra of $\mathbb{P}$-measurable sets is closed under κ-unions and -intersections. In particular, Borel sets are $\mathbb{P}$-measurable.

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In all practical cases $\mathbb{P}$ satisfies one of the above conditions.

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NB: This is completely analogous to the classical situation!

Σ^1_1

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Recall the **club filter** used by Halko & Shelah:

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$$C = \{x \in 2^\kappa \mid \{i < \kappa \mid x(i) = 1\} \text{ contains a club}\}.$$

For $S \subseteq \kappa$ stationary, co-stationary, define:

$$C_S = \{x \in \kappa^\kappa \mid \{i < \kappa \mid x(i) \in S\} \text{ contains a club}\}.$$

Clearly C_S is also $\mathbf{\Sigma}^1_1$.

Generalizing Halko-Shelah

Theorem (Friedman-Kh-Kulikov)

1. *If* $\mathbb{P}$ *is any tree-like forcing on* 2^κ **refining** $\mathbb{S}_\kappa$, *then* C *is not* $\mathbb{P}$*-measurable.*
2. *If* $\mathbb{P}$ *is any tree-like forcing on* κ^κ **refining** $\mathbb{M}_\kappa$, *then* C_S *is not* $\mathbb{P}$*-measurable.*

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Proof.

(1) Suppose C is $\mathbb{P}$-measurable, let $T \in \mathbb{P}$ be s.t. $[T] \subseteq^* C$ or $[T] \cap C =^* \varnothing$, w.l.o.g. the former. Let $\{X_i \mid i < \kappa\}$ be $\mathbb{P}$-nowhere dense sets such that $[T] \setminus C = \bigcup_{i<\kappa} X_i$. Construct a decreasing sequence of trees as follows:

- $T_0 := T$,
- $T_{i+1} \leq T_i$ is s.t. $[T_{i+1}] \cap X_i = \varnothing$ and $|\mathrm{stem}(T_{i+1})| > |\mathrm{stem}(T_i)|$,
- at limits λ, first let $T'_\lambda := \bigcap_{i<\lambda} T_i$, which is in $\mathbb{P}$ **by assumption**. Choose $T_\lambda \leq T'_\lambda$ such that $\mathrm{stem}(T_\lambda) \supseteq \mathrm{stem}(T'_\lambda)^\frown \langle 0 \rangle$.

Now $x := \bigcup_{i<\kappa} \mathrm{stem}(T_i)$ is a branch through T, $x \notin X_i$ for all $i < \kappa$, and $x(i) = 0$ **for club-many** $i < \kappa$, hence $x \notin C$—contradiction. $\square$

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Proof.

(2) Proceed analogously, except that at limit stages choose $T_\lambda \leq T'_\lambda$ such that $\mathrm{stem}(T_\lambda) \supseteq \mathrm{stem}(T'_\lambda)^\frown \langle\alpha\rangle$, where α is in S or not in S depending on what we want. $\square$

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Corollary

For all $\mathbb{P}$ refining $\mathbb{S}_\kappa$ or $\mathbb{M}_\kappa$, $\boldsymbol{\Sigma}^1_1$($\mathbb{P}$-measurability) is false.

$\boldsymbol{\Delta}^1_1$ sets

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In L, use the $\mathbf{\Sigma}^1_1$-good wellorder to construct counterexamples to $\mathbf{\Delta}^1_1(\mathbb{P})$, for any $\mathbb{P}$, by diagonalization.

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Question: Is $\mathbf{\Delta}^1_1(\mathbb{P}\text{-measurability})$ consistent?

Forcing $\mathbf{\Delta}^1_1$-$\mathbb{P}$-measurability

Theorem (Friedman-Kh-Kulikov)

Let $\mathbb{P}$ be a $<\kappa$-closed, κ-tree-like forcing.

1. *Suppose $\mathbb{P}$ satisfies the κ^+-c.c., and let $\mathbb{P}_{\kappa^+}$ be the κ^+-iteration of $\mathbb{P}$ with supports of size $<\kappa$. Then $V^{\mathbb{P}_{\kappa^+}} \models \mathbf{\Delta}^1_1(\mathbb{P}\text{-measurability})$.*
2. *Suppose $\mathbb{P}$ satisfies Axiom A^*, and let $\mathbb{P}_{\kappa^+}$ be the κ^+-iteration of $\mathbb{P}$ with supports of size $\leq\kappa$. Moreover, assume that for every $x \in \kappa^\kappa \cap V^{\mathbb{P}_{\kappa^+}}$, there is $\alpha < \kappa^+$ such that $x \in \kappa^\kappa \cap V^{\mathbb{P}_\alpha}$. Then $V^{\mathbb{P}_{\kappa^+}} \models \mathbf{\Delta}^1_1(\mathbb{P}\text{-measurability})$.*

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2. *Suppose $\mathbb{P}$ satisfies Axiom A^*, and let $\mathbb{P}_{\kappa^+}$ be the κ^+-iteration of $\mathbb{P}$ with supports of size $\leq\kappa$. Moreover, assume that for every $x \in \kappa^\kappa \cap V^{\mathbb{P}_{\kappa^+}}$, there is $\alpha < \kappa^+$ such that $x \in \kappa^\kappa \cap V^{\mathbb{P}_\alpha}$. Then $V^{\mathbb{P}_{\kappa^+}} \models \mathbf{\Delta}^1_1(\mathbb{P}\text{-measurability})$.*

All forcings we consider are $<\kappa$-closed and satisfy either the κ^+-c.c. or Axiom A*. However, the red condition is essentially about "preservation of κ-properness", which is a very difficult problem in the generalized setting.

Towards the proof

For the proof, we need a lemma which is proved similarly to the ω^ω-case.

Lemma

Let $\mathbb{P}$ be as in the theorem. For every elementary submodel $M \prec \mathcal{H}_\theta$ of a sufficiently large $\mathcal{H}_\theta$, with $|M| = \kappa$ and $M^{<\kappa} \subseteq M$, and for every $T \in \mathbb{P} \cap M$, there is $T' \leq T$ such that

$$[T'] \subseteq^* \{x \in \kappa^\kappa \mid x \text{ is } \mathbb{P}\text{-generic over } M\}.$$

(where $\subseteq^*$ means "modulo $\mathbb{P}$-meager" and a κ-real x is $\mathbb{P}$-generic over M iff $\{S \in \mathbb{P} \cap M \mid x \in [S]\}$ is a $\mathbb{P}$-generic filter over M.)

Proof of theorem

Proof.

In $V[G_{\kappa^+}]$, let A be $\mathbf{\Delta}^1_1$, defined by Σ^1_1-formulas ϕ and ψ. Let $S \in \mathbb{P}$ be arbitrary. We must find $T \leq S$ such that $[T] \subseteq^* A$ or $T \cap A =^* \varnothing$.

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By assumption, there exists $\alpha < \kappa^+$ s.t. S and the parameters of ϕ and ψ belong to $V[G_\alpha]$. Moreover, there is a $\beta > \alpha$ s.t. S belongs to $G(\beta+1)$. Let $x_{\beta+1}$ be the $(\beta+1)$-th generic real.

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In $V[G_{\kappa^+}]$, either $\phi(x_{\beta+1})$ or $\psi(x_{\beta+1})$ holds. By symmetry, we may w.l.o.g. assume the former. **Since (the iteration of) $\mathbb{P}$ is $<\kappa$-closed, we have Σ^1_1-absoluteness between $V[G_{\kappa^+}]$ and $V[G_{\beta+1}]$.** In particular, $V[G_{\beta+1}] \models \phi(x_{\beta+1})$. By the **forcing theorem** there exists $T \in V[G_\beta]$, $T \leq S$ and $T \Vdash_{\mathbb{P}} \phi(\dot{x}_{\mathrm{gen}})$.

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Take an elementary M of size κ containing T. By elementarity, $M \models$ "$T \Vdash_{\mathbb{P}} \phi(\dot{x}_{\mathrm{gen}})$". Going back to $V[G_{\kappa^+}]$, use the previous lemma to find $T' \leq T$ such that $[T'] \subseteq^* \{x \mid x$ is $\mathbb{P}$-generic over $M\}$. Now note that if x is $\mathbb{P}$-generic over M and $x \in [T]$, then $M[x] \models \phi(x)$. **By upwards-Σ^1_1-absoluteness** between M and $V[G_{\kappa^+}]$ we conclude that $\phi(x)$ really holds. Since this was true for arbitrary $x \in [T']$, we obtain $[T'] \subseteq^* \{x \mid \phi(x)\} = A$. $\square$

Independence for $\mathbf{\Delta}^1_1$ sets

Corollary

Let $\mathbb{P}$ be as in the assumption of the theorem. Then $\mathbf{\Delta}^1_1$($\mathbb{P}$-measurability) is independent.

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The proof of the above theorem is related to classical proofs for $\mathbf{\Delta}^1_2$ sets. So a natural question is: how much of the theory for classical $\mathbf{\Delta}^1_2$ sets holds for $\mathbf{\Delta}^1_1$ sets in the generalized context?

More on $\mathbf{\Delta}^1_1$.

4. Imitating classical $\mathbf{\Delta}^1_2$-theory for $\mathbf{\Delta}^1_1$-level?

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Theorem (Judah-Shelah 1989)

$\mathbf{\Delta}^1_2$*(Baire property) holds iff for every* $r \in \omega^\omega$ *there exists a Cohen real over* $L[r]$.

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Theorem (Judah-Shelah 1989)

$\mathbf{\Delta}^1_2$*(Baire property) holds iff for every* $r \in \omega^\omega$ *there exists a Cohen real over* $L[r]$.

Does this hold for $\mathbf{\Delta}^1_1$ sets in the generalized context?

More on $\mathbf{\Delta}^1_1$.

No!

Theorem (Friedman, Wu & Zdomskyy 2014)

Suppose κ is successor. There is a forcing iteration starting from L, in which cofinally many iterands have the κ^+-c.c., such that in the extension the club filter is $\mathbf{\Delta}^1_1$.

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One can verify that this iteration adds κ-Cohen reals cofinally often! Hence, in that model there are κ-Cohen reals over $L[r]$, for every $r \in 2^\kappa$, however $\mathbf{\Delta}^1_1$(κ-Baire property) fails.

More on $\boldsymbol{\Delta}^1_1$

Still, there are a few things we can say.

Fact

$\boldsymbol{\Delta}^1_1(\kappa\text{-Baire property}) \Rightarrow \forall r \in \kappa^\kappa\ \exists \kappa$-Cohen real over $L[r]$.

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Still, there are a few things we can say.

Fact

$\mathbf{\Delta}^1_1$(κ-Baire property) $\Rightarrow \forall r \in \kappa^\kappa \, \exists \kappa$-Cohen real over $L[r]$.

Lemma

Suppose κ inaccessible. Then $\mathbf{\Delta}^1_1(\mathbb{M}_\kappa$-measurability$) \Rightarrow \forall r \in \kappa^\kappa \, \exists x$ (x is unbounded over $L[r]$).

(This means $\{i \mid x(i) > y(i)\}$ is unbounded in κ, for every $y \in L[r]$).

Proof.

Based on the ω^ω-proof of Brendle & Löwe, but very technical. $\square$

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Still, there are a few things we can say.

Fact

$\mathbf{\Delta}^1_1(\kappa\text{-Baire property}) \Rightarrow \forall r \in \kappa^\kappa\ \exists \kappa$-Cohen real over $L[r]$.

Lemma

Suppose κ inaccessible. Then $\mathbf{\Delta}^1_1(\mathbb{M}_\kappa\text{-measurability}) \Rightarrow \forall r \in \kappa^\kappa\ \exists x\ (x$ is unbounded over $L[r])$.

(This means $\{i \mid x(i) > y(i)\}$ is unbounded in κ, for every $y \in L[r]$).

Proof.

Based on the ω^ω-proof of Brendle & Löwe, but very technical. □

We do not have any similar results for the other forcings notions.

Implication diagram on $\mathbf{\Delta}^1_1$ level

$$
\begin{tikzcd}
 & \mathbf{\Delta}^1_1(\mathbb{V}_\kappa) \arrow[rr, Rightarrow] & & \mathbf{\Delta}^1_1(\mathbb{S}_\kappa) \\
\mathbf{\Delta}^1_1(\mathbb{R}_\kappa) \arrow[ur, Rightarrow, dashed, "?"] \arrow[r, Rightarrow, dashed, "?"] \arrow[rr, Rightarrow, bend right] & \mathbf{\Delta}^1_1(\mathbb{L}_\kappa) \arrow[r, Rightarrow] & \mathbf{\Delta}^1_1(\mathbb{M}_\kappa) \arrow[ur, Rightarrow] & \\
 & & \mathbf{\Delta}^1_1(\mathbb{C}_\kappa) \arrow[u, Rightarrow] \arrow[uul, Rightarrow, "\kappa \text{ inacc.}"'] &
\end{tikzcd}
$$

$\mathbb{C}_\kappa$ = Cohen, $\mathbb{S}_\kappa$ = Sacks, $\mathbb{M}_\kappa$ = Miller, $\mathbb{L}_\kappa$ = Laver, $\mathbb{R}_\kappa$ = Mathias, $\mathbb{V}_\kappa$ = Silver.

The proofs are straightforward but quite technical.

Are the implications strict?

Can we prove that some/any of these implications are strict, i.e., cannot be reversed?

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Theorem (Friedman-Kh-Kulikov)

Suppose κ is inaccessible. Then $\mathrm{Con}(\mathbf{\Delta}^1_1(\mathbb{V}_\kappa\text{-measurability}) + \neg\mathbf{\Delta}^1_1(\mathbb{M}_\kappa))$.

Proof.

Perform a κ^+-iteration of κ-Silver forcing, starting in L, with supports of size κ. Then $\mathbf{\Delta}^1_1(\mathbb{V}_\kappa$-measurability) holds by our previous theorem. Next, show that "κ-properness" is preserved (similar to Kanamori's κ-Sacks). Using inaccessibility of κ, the iteration is "κ^κ-bounding". As a result, the generic extension does not satisfy the statement "$\forall r\, \exists x$ (x is unbounded over $\kappa^\kappa \cap L[r]$)", so $\mathbf{\Delta}^1_1(\mathbb{M}_\kappa$-measurability) fails. $\square$

Implication diagram on $\mathbf{\Delta}^1_1$ level

However, there are still **many** open questions!

$$\begin{tikzcd}
 & \mathbf{\Delta}^1_1(\mathbb{V}_\kappa) \arrow[rr, Rightarrow] & & \mathbf{\Delta}^1_1(\mathbb{S}_\kappa) \\
\mathbf{\Delta}^1_1(\mathbb{R}_\kappa) \arrow[ur, Rightarrow, dashed, "?"] \arrow[r, Rightarrow, dashed, "?"] \arrow[rr, Rightarrow, bend right] & \mathbf{\Delta}^1_1(\mathbb{L}_\kappa) \arrow[r, Rightarrow] & \mathbf{\Delta}^1_1(\mathbb{M}_\kappa) \arrow[ur, Rightarrow] & \\
 & & \mathbf{\Delta}^1_1(\mathbb{C}_\kappa) \arrow[u, Rightarrow] \arrow[uul, Rightarrow, "\kappa \text{ inacc.}"'] &
\end{tikzcd}$$

$\mathbb{C}_\kappa$ = Cohen, $\mathbb{S}_\kappa$ = Sacks, $\mathbb{M}_\kappa$ = Miller, $\mathbb{L}_\kappa$ = Laver, $\mathbb{R}_\kappa$ = Mathias, $\mathbb{V}_\kappa$ = Silver.

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All the properties we looked at are determined by **forcing** posets. We want these forcings to be $<\kappa$-**closed**, so the trees $T \in \mathbb{P}$ are required to have a certain shape.

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All the properties we looked at are determined by **forcing** posets. We want these forcings to be $<\kappa$-**closed**, so the trees $T \in \mathbb{P}$ are required to have a certain shape.

In particular, all our trees T satisfy:

$$\forall x \in [T] \, (\{i < \kappa \mid x{\restriction}i \text{ is a split-node of } T\} \text{ is club}).$$

What if we drop this property?

Some results

1. If we drop the assumption on κ-Sacks trees that "limits of split-nodes are split-nodes", we obtain a propery weaker than $\mathbb{S}_\kappa$-measurability, **which consistently holds for all generalized projective sets**, (Schlicht).

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2. If we drop the assumption on κ-Miller trees that "limits of club-splitting-nodes are club-splitting", we obtain a propery weaker than $\mathbb{M}_\kappa$-measurability, **which consistently holds for all generalized projective sets** (Laguzzi, independently Lücke-Motto Ros-Schlicht).

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3. If we drop the assumption on κ-Silver trees that "splitting levels from a club" and replace it by "splitting levels form a stationary set", we obtain a property weaker than $\mathbb{V}_\kappa$-measurability, **which consistently holds for all generalized projective sets** (Laguzzi).

Even more results

These weaker properties are not useful for forcing theory, because the corresponding forcing notions are **not $<\kappa$-closed.**

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Work in progress (Friedman & Laguzzi)

Assume κ is measurable. Consider a version of Silver forcing in which the trees are required to split **on a set positive with respect to a normal measure on** κ. The corresponding forcing is κ-proper and $<\kappa$-closed, and the corresponding regularity property is consistent for all projective sets.

Thank you!

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