Cichoń's Diagram and Regularity Properties

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Cichoń's diagram

"What pentagram is to heavy metal, Cichoń's diagam is to set theory."

— Jindřich Zapletal

$$\begin{tikzcd}
& \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
& & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
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1. Each inequality appearing in the diagram is provable in ZFC.
2. Each inequality **not** appearing in the diagram is **not** provable in ZFC, except
3. $\mathrm{add}(\mathcal{M}) = \min(\mathfrak{b}, \mathrm{cov}(\mathcal{M}))$ and $\mathrm{cof}(\mathcal{M}) = \max(\mathfrak{d}, \mathrm{non}(\mathcal{M}))$.

Regularity properties

Let $A \subseteq \omega^\omega$ or 2^ω.

- A has the **Baire property** iff for every basic open $[s]$ there is a basic open $[t] \subseteq [s]$ such that $[t] \subseteq^* A$ or $[t] \cap A =^* \varnothing$.
- A is **Lebesgue-measurable** iff for every closed set C of positive measure there is a closed subset $C' \subseteq C$ of positive measure, such that $C \subseteq A$ or $C \cap A = \varnothing$.

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Baire property $=$ Cohen forcing

Lebesgue measure $=$ random forcing

More Forcing Notions

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- $\mathbb{M}$ = Miller forcing: conditions are **super-perfect trees** on $\omega^{<\omega}$

  (every node has an extensions which is infinitely splitting).
- $\mathbb{L}$ = Laver forcing: conditions are **Laver trees**

  (every node beyond the stem is infinitely splitting).

More Regularity Properties

Definition

$A \subseteq 2^\omega$ is **Sacks-measurable** (Marczewski-measurable) iff

$$\forall T \in \mathbb{S}\, \exists S \in \mathbb{S}, S \leq T\ ([S] \subseteq A \text{ or } [S] \cap A = \varnothing).$$

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$A \subseteq \omega^\omega$ is **Laver-measurable** iff

$$\forall T \in \mathbb{L}\, \exists S \in \mathbb{L}, S \leq T\ ([S] \subseteq A \text{ or } [S] \cap A = \varnothing).$$

Regularity of projective sets

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$\boldsymbol{\Sigma}^1_1(\mathbb{P})$ is true.

But $\boldsymbol{\Sigma}^1_2(\mathbb{P})$ and $\boldsymbol{\Delta}^1_2(\mathbb{P})$ are already independent of ZFC.

Cichoń's diagram for regularity properties

$$
\begin{tikzcd}
 & \mathbf{\Delta}^1_2(\mathbb{B}) \arrow[rrr, Rightarrow] & & & \begin{array}{c}\mathbf{\Sigma}^1_2(\mathbb{S}) \\ \mathbf{\Delta}^1_2(\mathbb{S})\end{array} \\
 & & \begin{array}{c}\mathbf{\Sigma}^1_2(\mathbb{L}) \\ \mathbf{\Delta}^1_2(\mathbb{L})\end{array} \arrow[r, Rightarrow] & \begin{array}{c}\mathbf{\Sigma}^1_2(\mathbb{M}) \\ \mathbf{\Delta}^1_2(\mathbb{M})\end{array} \arrow[ur, Rightarrow] & \\
\forall r(\omega_1^{L[r]} < \omega_1) \arrow[r, Rightarrow] & \mathbf{\Sigma}^1_2(\mathbb{B}) \arrow[r, Rightarrow] \arrow[uu, Rightarrow] & \mathbf{\Sigma}^1_2(\mathbb{C}) \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \mathbf{\Delta}^1_2(\mathbb{C}) \arrow[u, Rightarrow] &
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1. Each implication appearing in the diagram is provable in ZFC.
2. Each implication **not** appearing in the diagram is **not** provable in ZFC, except
3. $\mathbf{\Delta}^1_2(\mathbb{L}) + \mathbf{\Delta}^1_2(\mathbb{C}) \implies \mathbf{\Sigma}^1_2(\mathbb{C})$

Characterization (1)

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Theorem (Judah-Shelah 1989)

The following are equivalent:

1. $\mathbf{\Delta}^1_2(\mathbb{C})$
2. $\forall r\, \exists x\, (x$ *is Cohen over* $L[r])$.

Theorem (Solovay 1970)

The following are equivalent:

1. $\mathbf{\Sigma}^1_2(\mathbb{C})$
2. $\forall r\, \{x \mid x$ *Cohen over* $L[r]\}$ *is comeager.*

Characterization (2)

Why this analogy?

Theorem (Judah-Shelah 1989)

The following are equivalent:

1. $\mathbf{\Delta}^1_2(\mathbb{B})$
2. $\forall r\, \exists x\, (x \text{ is random over } L[r])$.

Theorem (Solovay 1970)

The following are equivalent:

1. $\mathbf{\Sigma}^1_2(\mathbb{B})$
2. $\forall r\, \mu(\{x \mid x \text{ random over } L[r]\}) = 1$.

Characterization (3)

Theorem (Brendle-Löwe 1999)

- *The following are equivalent:*
  1. $\boldsymbol{\Delta}^1_2(\mathbb{L})$
  2. $\boldsymbol{\Sigma}^1_2(\mathbb{L})$
  3. $\forall r\, \exists x\, (x \text{ is dominating over } L[r])$
- *The following are equivalent:*
  1. $\boldsymbol{\Delta}^1_2(\mathbb{M})$
  2. $\boldsymbol{\Sigma}^1_2(\mathbb{M})$
  3. $\forall r\, \exists x\, (x \text{ is unbounded over } L[r])$
- *The following are equivalent:*
  1. $\boldsymbol{\Delta}^1_2(\mathbb{S})$
  2. $\boldsymbol{\Sigma}^1_2(\mathbb{S})$
  3. $\forall r\, \exists x\, (x \notin L[r])$

Correspondence

Regularity hypothesis	Transcendence over $L[r]$	Cardinal characteristic
$\forall r(\omega_1^{L[r]} < \omega_1)$	"making ground-model reals countable"	$\aleph_1$
$\mathbf{\Sigma}^1_2(\mathbb{B})$	measure-one many random reals	$\mathrm{add}(\mathcal{N})$
$\mathbf{\Delta}^1_2(\mathbb{B})$	random reals	$\mathrm{cov}(\mathcal{N})$
$\mathbf{\Sigma}^1_2(\mathbb{C})$	co-meager many Cohen reals	$\mathrm{add}(\mathcal{M})$
$\mathbf{\Delta}^1_2(\mathbb{C})$	Cohen reals	$\mathrm{cov}(\mathcal{M})$
$\mathbf{\Delta}^1_2(\mathbb{L})$ / $\mathbf{\Sigma}^1_2(\mathbb{L})$	dominating reals	$\mathfrak{b}$
$\mathbf{\Delta}^1_2(\mathbb{M})$ / $\mathbf{\Sigma}^1_2(\mathbb{M})$	unbounded reals	$\mathfrak{d}$
$\mathbf{\Delta}^1_2(\mathbb{S})$ / $\mathbf{\Sigma}^1_2(\mathbb{S})$	new reals	$2^{\aleph_0}$

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Analogy between hypotheses about regularity on 2nd level and cardinal characteristics.

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Analogy between hypotheses about regularity on 2nd level and cardinal characteristics.

Question

What happens at higher levels of the projective hierarchy?

Large cardinals?

One possible answer (Ikegami, Judah-Spinas, Friedman): assume suitable large cardinals to "lift" characterization theorems to higher levels, replacing L by some other suitable inner model.

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Then by similar arguments, one obtains the analogous diagram on higher projective levels.

Disadvantage of large cardinals

Our approach is different, for the following reasons:

1. Since **Proj**($\mathbb{P}$) can be obtained just from an inaccessible, it seems unnatural to require stronger hypotheses for questions about $\boldsymbol{\Sigma}^1_n(\mathbb{P})$ and $\boldsymbol{\Delta}^1_n(\mathbb{P})$ for **low** values of n (Bagaria, Judah, Shelah).

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2. Assuming **too** strong hypotheses trivializes the question (e.g. PD). We need **exactly the right** large cardinal strength, which seems artificial.

Additional motivation

Recall the Bartoszynski-Raisonnier-Stern implication: $\mathbf{\Sigma}^1_2(\mathbb{B}) \Rightarrow \mathbf{\Sigma}^1_2(\mathbb{C})$.

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We will see more examples.

ZFC

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Without **characterization results**, can anything at all be said for regularity properties on higher projective levels?

Some straightforward implications

Fact

Let $\mathbf{\Gamma}$ be **any** pointclass closed under continuous pre-images. Then:

1. $\mathbf{\Gamma}(\mathbb{L}) \Rightarrow \mathbf{\Gamma}(\mathbb{M}) \Rightarrow \mathbf{\Gamma}(\mathbb{S})$.
2. $\mathbf{\Gamma}(\mathbb{B}) \Rightarrow \mathbf{\Gamma}(\mathbb{S})$.
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Proof.

First, note that for $T \in \mathbb{P}$ there is a homemorphism between $[T]$ and ω^ω (or 2^ω). So we can ignore the "below any condition"-clause in the definition of $\mathbb{P}$-measurability!

1. A Laver tree is a Miller tree, which is (almost) a Sacks tree.
2. A closed set of positive measure contains a perfect subset of positive measure.
3. A set comeager in a basic open set contains a super-perfect tree. $\square$

Cichoń's diagram on the third level

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\end{tikzcd}$$

Eventually, we would like to "solve" this diagram in ZFC or ZFC + inaccessible.

Cohen and Random

Concerning Cohen and Random, some things were known:

- Con($\mathbf{\Delta}^1_3(\mathbb{C}) + \neg\mathbf{\Delta}^1_3(\mathbb{B})$) from ZFC (Bagaria-Judah 1993)

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- Con($\mathbf{\Delta}^1_3(\mathbb{C}) + \neg\mathbf{\Sigma}^1_3(\mathbb{C})$) and Con($\mathbf{\Delta}^1_3(\mathbb{B}) + \neg\mathbf{\Sigma}^1_3(\mathbb{B})$) from ZFC (Bagaria-Judah 1993)

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- Con($\mathbf{\Delta}^1_4(\mathbb{B}) + \neg\mathbf{\Delta}^1_4(\mathbb{C})$) from inaccessible (Judah-Spinas 1995)

Solving the diagrams

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Are $\mathbf{\Sigma}^1_3(\mathbb{P})$ and $\mathbf{\Delta}^1_3(\mathbb{P})$ equivalent for $\mathbb{P} \in \{\mathbb{S}, \mathbb{L}, \mathbb{M}\}$?

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But it is easier if we restrict attention exclusively to $\boldsymbol{\Delta}^1_3$, $\boldsymbol{\Sigma}^1_3$ or $\boldsymbol{\Delta}^1_4$ sets!

The $\mathbf{\Delta}^1_3$-diagram

$$\begin{tikzcd}
\mathbf{\Delta}^1_3(\mathbb{B}) \arrow[rr, Rightarrow] & & \mathbf{\Delta}^1_3(\mathbb{S}) \\
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\end{tikzcd}$$

There are 11 possible combinations of assigning "true" and "false" to these regularity statements without contradicting the diagram.

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There are 11 possible combinations of assigning "true" and "false" to these regularity statements without contradicting the diagram.

Theorem (Fischer-Friedman-Kh)

Using ZFC + inaccessible, there is a model for each such combination.

In 8 out of 11 cases, ZFC is sufficient.

The Σ^1_3-diagram

$$\begin{tikzcd}
\mathbf{\Sigma}^1_3(\mathbb{B}) \arrow[rr, Rightarrow] & & \mathbf{\Sigma}^1_3(\mathbb{S}) \\
\mathbf{\Sigma}^1_3(\mathbb{L}) \arrow[r, Rightarrow] & \mathbf{\Sigma}^1_3(\mathbb{M}) \arrow[ur, Rightarrow] & \\
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From ZFC + inaccessible, 5 out of 11 combinations have a model.

The $\mathbf{\Delta}^1_4$-diagram

$$\begin{tikzcd}
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Theorem (Fischer-Friedman-Kh)

From ZFC + inaccessible, 7 out of 11 combinations have a model.

Separating Σ from Δ

Theorem (Fischer-Friedman-Kh)

Starting from ZFC, there is a model where $\mathbf{\Delta}^1_3(\mathbb{P})$ *holds for all* $\mathbb{P}$ *but* $\mathbf{\Sigma}^1_3(\mathbb{B})$ *and* $\mathbf{\Sigma}^1_3(\mathbb{C})$ *fail.*

Starting from ZFC + inaccessible, there is a model where $\mathbf{\Delta}^1_4(\mathbb{P})$ *holds for all* $\mathbb{P}$ *but* $\mathbf{\Sigma}^1_4(\mathbb{B})$ *and* $\mathbf{\Sigma}^1_4(\mathbb{C})$ *fail.*

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Recall the Truss-implication: $\mathbf{\Delta}^1_2(\mathbb{L}) + \mathbf{\Delta}^1_2(\mathbb{C}) \Rightarrow \mathbf{\Sigma}^1_2(\mathbb{C})$.

By the above theorem, this fails to lift to the 3rd and 4th levels.

The main ingredients

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All our results are obtained by ω_1-iterations of Suslin or Suslin$^+$ proper forcing, with countable support, starting from L, L^* or L^d. If we use L or L^* we have a ZFC-proof; if we use L^d we require an inaccessible.

For the rest of the talk...

1. Suslin and Suslin$^+$ proper forcing.
2. Methods for obtaining regularity.
3. Solving the diagrams.

1. Suslin and Suslin$^+$ proper forcing.

Properness Without Elementaricity

Suslin and Suslin$^+$ proper forcing (in general "non-elementary proper forcing") has been studied by Judah, Shelah, Goldstern and Kellner.

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- Idea: replace $M \prec \mathcal{H}_\theta$ by **any** countable transitive model M of (a sufficient fragment of) ZFC.
- But "$\mathbb{P} \cap M$" etc. does not make sense when M is not elementary.

Suslin proper forcing

Definition

A forcing notion $\mathbb{P}$ is **Suslin** if elements of $\mathbb{P}$ are (coded by) reals and "$p \in \mathbb{P}$", "$p \leq q$" and "$p \perp q$" are $\mathbf{\Sigma}^1_1$ relations.

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Suslin$^+$ proper forcing

Problem: Unfortunately, many standard forcing notions (in particular Sacks, Miller and Laver) are not **exactly** Suslin, because $\perp$ is only $\mathbf{\Pi}^1_1$ but not $\mathbf{\Sigma}^1_1$.

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Solution: (Shelah; Goldstern) Replace "Suslin" by "Suslin$^+$", where we don't require $\perp$ to be $\mathbf{\Sigma}^1_1$. Instead, we make sure that there is an "effective" version of being an $(M, \mathbb{P})$-generic condition.

Technically, require that there exists a $\mathbf{\Sigma}^1_2$, $(\omega+1)$-place relation $\mathrm{epd}(p_i, q)$ such that if $\mathrm{epd}(p_i, q)$ holds then $\{p_i \mid i < \omega\}$ is predense below q (provably in ZFC), and use epd to define an effectively $(M, \mathbb{P})$-generic condition.

Remarks about Suslin$^+$

Remarks:

1. All standard definable forcings used in the theory of the reals which are known to be proper, are actually Suslin$^+$ proper.
2. In fact, they satisfy an effective version of Axiom A which implies Suslin$^+$ properness (Kellner 2006).

Why is this useful?

Iterations of Suslin$^+$ proper forcing notions satisfy nice properties which other iterations do not.

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3. If $V \models \forall r\,(\omega_1^{L[r]} < \omega_1)$ then $V^{\mathbb{P}_\alpha} \models \forall r\,(\omega_1^{L[r]} < \omega_1)$.
4. If $V \models \forall r\,(\omega_1^{L[r]} < \omega_1)$ then $\mathbf{\Sigma}^1_3$-absoluteness holds **between any pair of models** N and N' with $V \subseteq N \subseteq N' \subseteq V^{\mathbb{P}_\alpha}$.

   (This was proved by Judah for Suslin ccc forcing).

2. Methods for obtaining regularity.

Methods for obtaining regularity

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Classical results say, roughly:

- An iteration of length ω_1 of $\mathbb{P}$ yields $\boldsymbol{\Delta}^1_2(\mathbb{P})$, and
- An iteration of length ω_1 of "amoeba-for-$\mathbb{P}$" yields $\boldsymbol{\Sigma}^1_2(\mathbb{P})$.

The point is to squeeze out stronger results using Suslin$^+$ properness.

Amoeba and Quasi-amoeba

Definition

Let $\mathbb{P}$ be a tree-like forcing notion, and $\mathbb{AP}$ another forcing. We say that

1. $\mathbb{AP}$ is a **quasi-amoeba for** $\mathbb{P}$ if for every $p \in \mathbb{P}$ and every $\mathbb{AP}$-generic G, in $V[G]$ there is a $q \leq p$ such that

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2. $\mathbb{AP}$ is an **amoeba for** $\mathbb{P}$ if for every $p \in \mathbb{P}$ and every $\mathbb{AP}$-generic G, in $V[G]$ there is a $q \leq p$ such that for any larger model $W \supseteq V[G]$,
$$W \models \forall x \in [q]\,(x \text{ is } \mathbb{P}\text{-generic over } V).$$

Examples

For Cohen and random, **quasi-amoeba** and **amoeba** are the same thing. But in general they are different.

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Examples:

1. $\mathbb{S}$ is a quasi-amoeba, but not an amoeba, for itself (Brendle 1998).
2. $\mathbb{M}$ is a quasi-amoeba, but not an amoeba, for itself (Brendle 1998).
3. $\mathbb{L}$ is **not** a quasi-amoeba for itself (Brendle 1998), but there are amoebas for $\mathbb{L}$.
4. Mathias forcing $\mathbb{R}$ is an amoeba for itself.

The methods

Method 1 (Bagaria-Judah)

1. If $V \models \boldsymbol{\Sigma}^1_2(\mathbb{B})$ then $V^{\mathbb{B}_{\omega_1}} \models \boldsymbol{\Delta}^1_3(\mathbb{B})$.
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Suppose $\mathbb{AP}_i$ is a quasi-amoeba for $\mathbb{P}_i$ for all $i \leq k$, and all $\mathbb{P}_i$ and $\mathbb{AP}_i$ are Suslin$^+$ proper. Then $V^{(\mathbb{P}_0 * \mathbb{AP}_0 * \cdots * \mathbb{P}_k * \mathbb{AP}_k)_{\omega_1}} \models \boldsymbol{\Delta}^1_3(\mathbb{P}_i)$ for each i.

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Corollary

$V^{\mathbb{S}_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{S})$ and $V^{\mathbb{M}_{\omega_1}} \models \mathbf{\Delta}^1_3(\mathbb{M})$.

Some aspects of the proof

This method is based on the proof of Judah-Shelah that the ω_1-iteration of Mathias-forcing yields $\boldsymbol{\Delta}^1_3$(Ramsey).

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Idea: in the intermediary extension, the quasi-amoeba gives us a $\mathbb{P}$-condition q such that all $x \in [q]$ are generic over the ground model. From this we conclude (in this intermediary extension) that for all $x \in [q]$, the statement "a certain condition forces a certain $\mathbf{\Pi}^1_2$-statement concerning x" is true. Then in the final extension, it may **not** be true that all $x \in [q]$ are generic but that **doesn't matter** because the above statement is preserved by $\mathbf{\Pi}^1_2$-absoluteness.

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For this to work, we rely heavily on properties of Suslin$^+$ proper iterations!

More methods

Method 3 (Fischer-Friedman-Kh)

Suppose $V \models \forall r\, (\omega_1^{L[r]} < \omega_1)$ and $\mathbb{P}_{\omega_1} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1 \rangle$ is an iteration of Suslin$^+$ proper forcing notions in which $\mathbb{P}$ appears cofinally often. Then $V^{\mathbb{P}_{\omega_1}} \models \boldsymbol{\Delta}^1_3(\mathbb{P})$.

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Idea: Here, we use the $\mathbf{\Sigma}^1_3$-absoluteness between models N, N' with $V \subseteq N \subseteq N' \subseteq V[G_{\omega_1}]$, and the preservation of $\forall r\, (\omega_1^{L[r]} < \omega_1)$ by Suslin$^+$ proper iterations.

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Method 4 (Fischer-Friedman-Kh)

Suppose $V \models \forall r\,(\omega_1^{L[r]} < \omega_1)$, $\mathbb{AP}_i$ is a quasi-amoeba for $\mathbb{P}_i$ for all $i \leq k$, and all $\mathbb{P}_i$ and $\mathbb{AP}_i$ are Suslin$^+$ proper. Then $V^{(\mathbb{P}_0 * \mathbb{AP}_0 * \cdots * \mathbb{P}_k * \mathbb{AP}_k)_{\omega_1}} \models \mathbf{\Delta}^1_4(\mathbb{P}_i)$ for each i.

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Idea: Direct generalization of Method 2, using $\boldsymbol{\Sigma}^1_3$-absoluteness between intermediary models instead of Shoenfield absoluteness.

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3. Solving the diagrams.

Separation

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Fact

Let V be a model with a Σ^1_3-good wellorder of the reals.

1. If there are no Cohen reals over V then $\neg\boldsymbol{\Delta}^1_3(\mathbb{C})$.
2. If there are no random reals over V then $\neg\boldsymbol{\Delta}^1_3(\mathbb{B})$.
3. If there are no dominating reals over V then $\neg\boldsymbol{\Delta}^1_3(\mathbb{L})$.
4. If there are no unbounded reals over V then $\neg\boldsymbol{\Delta}^1_3(\mathbb{M})$.
5. If $\omega^\omega \cap V = \omega^\omega$ then $\neg\boldsymbol{\Delta}^1_3(\mathbb{S})$.

Solving the $\mathbf{\Delta}^1_3$-diagram

$$
\begin{tikzcd}
\mathbf{\Delta}^1_3(\mathbb{B}) \arrow[rrr, Rightarrow] & & & \mathbf{\Delta}^1_3(\mathbb{S}) \\
& \mathbf{\Delta}^1_3(\mathbb{L}) \arrow[r, Rightarrow] & \mathbf{\Delta}^1_3(\mathbb{M}) \arrow[ur, Rightarrow] & \\
& & \mathbf{\Delta}^1_3(\mathbb{C}) \arrow[u, Rightarrow] &
\end{tikzcd}
$$

Recall:

1. L^* is a model where
   - $\mathbf{\Sigma}^1_2(\mathbb{P})$ holds for all $\mathbb{P}$, but
   - there is a Σ^1_3-good wellorder of the reals.
2. L^d is a model where
   - $\forall r\, (\omega_1^{L[r]} < \omega_1)$, but
   - there is a Σ^1_3-good wellorder of the reals.

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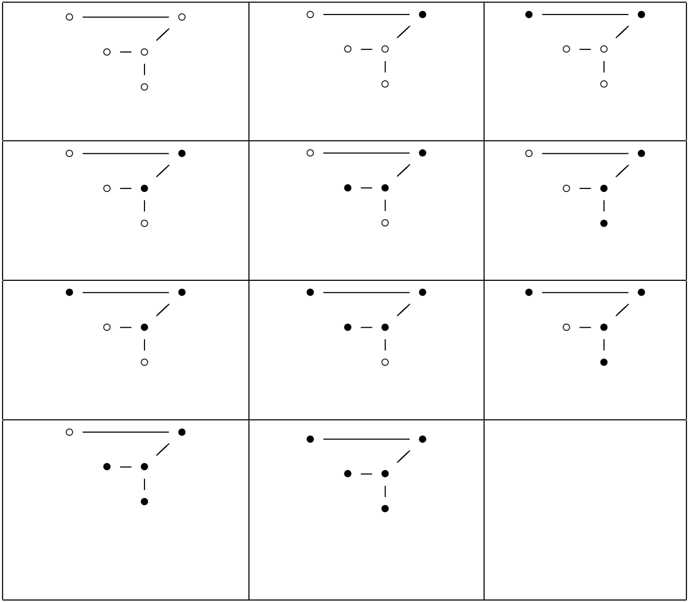

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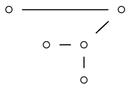
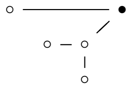
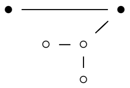
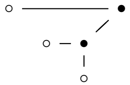
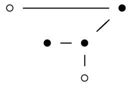
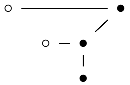
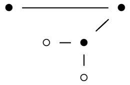
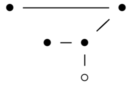
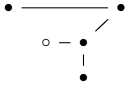
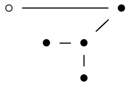
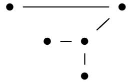
 L	 $L^{\mathbb{S}_{\omega_1}}$	 $(L^*)^{\mathbb{B}_{\omega_1}}$
 $L^{\mathbb{M}_{\omega_1}}$	 $L^{(\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $L^{\mathbb{R}_{\omega_1}}$	 $(L^*)^{\mathbb{C}_{\omega_1}}$
 $(L^d)^{(\mathbb{B}*\mathbb{M})_{\omega_1}}$	 $(L^d)^{(\mathbb{B}*\mathbb{L})_{\omega_1}}$	 $(L^d)^{(\mathbb{B}*\mathbb{C})_{\omega_1}}$
 $(L^d)^{(\mathbb{C}*\mathbb{L})_{\omega_1}}$ or a ZFC-model of Bartoszyński-Judah	 $L^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $L^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{R})_{\omega_1}}$	

$$\begin{tikzcd}
\boldsymbol{\Sigma}^1_3(\mathbb{B}) \arrow[rr, Rightarrow] & & \boldsymbol{\Sigma}^1_3(\mathbb{S}) \\
\boldsymbol{\Sigma}^1_3(\mathbb{L}) \arrow[r, Rightarrow] & \boldsymbol{\Sigma}^1_3(\mathbb{M}) \arrow[ur, Rightarrow] & \\
& \boldsymbol{\Sigma}^1_3(\mathbb{C}) \arrow[u, Rightarrow] &
\end{tikzcd}$$

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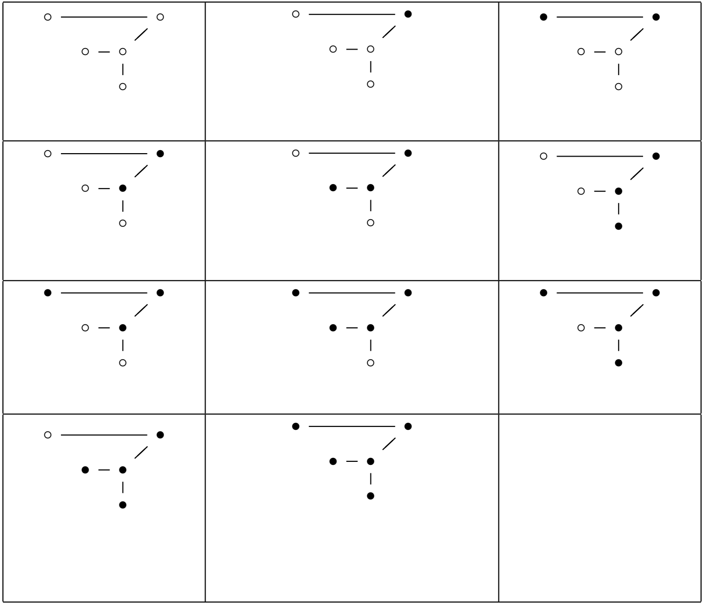

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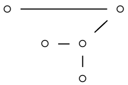
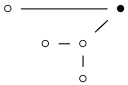
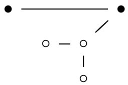
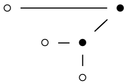
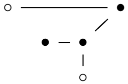
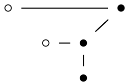
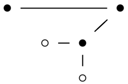
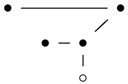
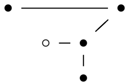
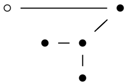
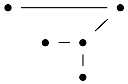
 L	 $(L^d)^{\mathbb{S}_{\omega_1}}$	 ???
 $(L^d)^{\mathbb{M}_{\omega_1}}$	 $(L^d)^{(\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $(L^d)^{\mathbb{R}_{\omega_1}}$	 ???
 ???	 ???	 ???
 ???	 $(L^d)^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $(L^d)^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{R})_{\omega_1}}$ (or Solovay Model)	

$$
\begin{tikzcd}
\boldsymbol{\Delta}^1_4(\mathbb{B}) \arrow[rrr, Rightarrow] & & & \boldsymbol{\Delta}^1_4(\mathbb{S}) \\
& \boldsymbol{\Delta}^1_4(\mathbb{L}) \arrow[r, Rightarrow] & \boldsymbol{\Delta}^1_4(\mathbb{M}) \arrow[ur, Rightarrow] & \\
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\end{tikzcd}
$$

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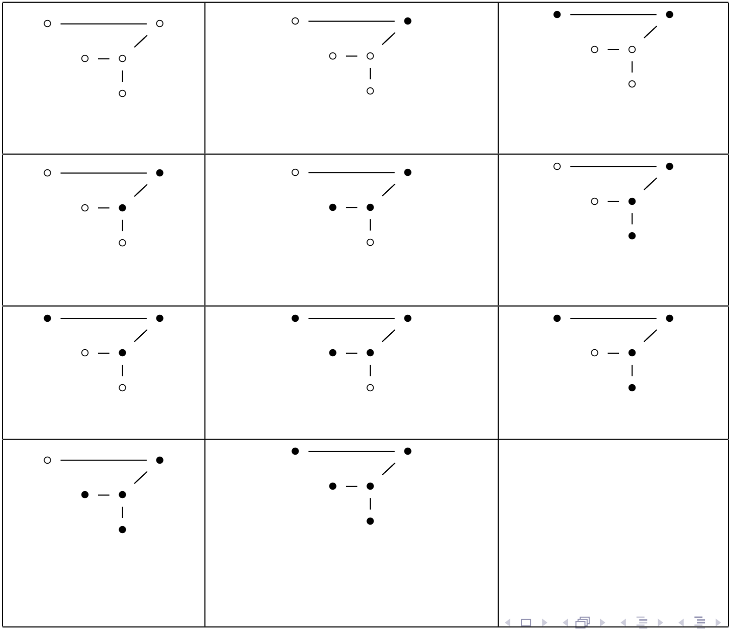

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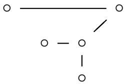
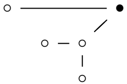
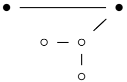
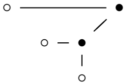
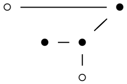
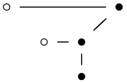
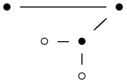
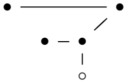
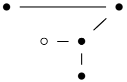
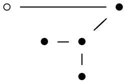
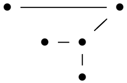
 L	 $(L^d)^{\mathbb{S}_{\omega_1}}$	 ???
 $(L^d)^{\mathbb{M}_{\omega_1}}$	 $(L^d)^{(\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $(L^d)^{\mathbb{R}_{\omega_1}}$	 ???
 ???	 ???	 ???
 ???	 $(L^d)^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $(L^d)^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{R})_{\omega_1}}$ (or Solovay Model)	

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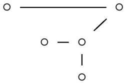
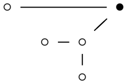
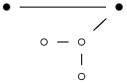
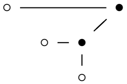
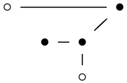
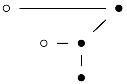
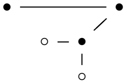
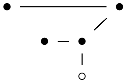
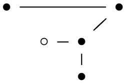
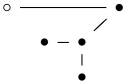
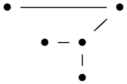
 L	 $(L^d)^{\mathbb{S}_{\omega_1}}$	 Judah-Spinas 1995
 $(L^d)^{\mathbb{M}_{\omega_1}}$	 $(L^d)^{(\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $(L^d)^{\mathbb{R}_{\omega_1}}$	 Judah-Spinas 1995
 ???	 ???	 ???
 ???	 $(L^d)^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{L}*\mathbb{AL})_{\omega_1}}$ or $(L^d)^{(\mathbb{B}*\mathbb{A}*\mathbb{C}*\mathbb{R})_{\omega_1}}$ (or Solovay Model)	

Separating Δ from Σ

Theorem (Fischer-Friedman-Kh)

Starting from ZFC, there is a model where $\mathbf{\Delta}^1_3(\mathbb{P})$ *holds for all* $\mathbb{P}$ *but* $\mathbf{\Sigma}^1_3(\mathbb{B})$ *and* $\mathbf{\Sigma}^1_3(\mathbb{C})$ *fail.*

Starting from ZFC + inaccessible, there is a model where $\mathbf{\Delta}^1_4(\mathbb{P})$ *holds for all* $\mathbb{P}$ *but* $\mathbf{\Sigma}^1_4(\mathbb{B})$ *and* $\mathbf{\Sigma}^1_4(\mathbb{C})$ *fail.*

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Idea:

- For the first assertion, use the $\mathbf{\Sigma}^1_3$ **Raisonnier filter** defined from the reals of L, and $\omega_1^L = \omega_1$.
- For the second assertion, use the $\mathbf{\Sigma}^1_4$ **Raisonnier filter** defined using the reals of L^d and $\omega_1^{L^d} = \omega_1$.

Open Questions

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1. Is $\mathbf{\Sigma}^1_3(\mathbb{P})$ and $\mathbf{\Delta}^1_3(\mathbb{P})$ equivalent for Sacks, Miller and Laver? (we conjecture that they are not).

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1. Is $\mathbf{\Sigma}^1_3(\mathbb{P})$ and $\mathbf{\Delta}^1_3(\mathbb{P})$ equivalent for Sacks, Miller and Laver? (we conjecture that they are not).
2. Solve the $\mathbf{\Delta}^1_3$-diagram in ZFC.
3. Solve the other diagrams.
4. Consistency strength of $\mathbf{\Sigma}^1_3(\mathbb{L})$?

Thank you!

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