A General Setting for the Pointwise Investigation of Determinacy

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Games in Set Theory

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The **Axiom of Determinacy** says "every set of reals is determined".

Axiom of Determinacy

- AD contradicts the Axiom of Choice,
- AD $\rightarrow$ all sets of reals are Lebesgue-measurable,
- AD $\rightarrow$ all sets of reals have the Baire property,
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Question: is it true that "A is determined" $\rightarrow$ "A is regular"?

Class-wise implication

No, because the games used involve **coding**. But if Γ is a collection of sets closed under some natural operations, then

$$\begin{array}{c}\text{Every set in } \Gamma \\ \text{is determined}\end{array} \Rightarrow \begin{array}{c}\text{every set in } \Gamma \\ \text{is regular}\end{array}$$

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Example: $\boldsymbol{\Gamma} \subseteq$ **Det** $\rightarrow \boldsymbol{\Gamma} \subseteq$ BP.

Proof:

- Define the Banach-Mazur game, G^{**}.
- Encode $A \rightsquigarrow A'$ so that $G^{**}(A) \equiv G(A')$.
- Then: I wins $G(A') \iff A$ is comeager in an open set
  II wins $G(A') \iff A$ is meager.
- If $A \in \boldsymbol{\Gamma}$ then $A' \in \boldsymbol{\Gamma}$ so $G(A')$ is determined. Then A is either comeager in an open set or meager.
- If all sets in $\boldsymbol{\Gamma}$ have this property, then all sets in $\boldsymbol{\Gamma}$ have the Baire property. $\square$

Point-wise implication

Benedikt Löwe: What is the strength of the statement “A is determined”?

The pointwise view of determinacy: arboreal forcings, measurability, and weak measurability, Rocky Mountains Journal of Mathematics **35** (2005)

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(AC) Sets can be **determined** but not **regular**.

Setting used: Arboreal forcing notions and their algebras of measurability.

Arboreal Forcings

Definition:

- **Arboreal forcing:** a partial order $(\mathbb{P}, \leq)$ of trees (closed sets of reals) on ω or 2 ordered by inclusion, and

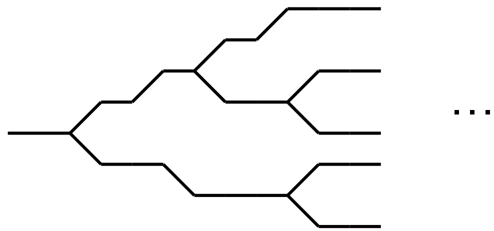
$$\forall P \in \mathbb{P}\, \forall t \in P\ (P{\uparrow}t \in \mathbb{P})$$

- An arboreal $(\mathbb{P}, \leq)$ is called **topological** if $\{[P] \mid P \in \mathbb{P}\}$ is a topology base on ω^ω or 2^ω. Otherwise, it is called **non-topological**.

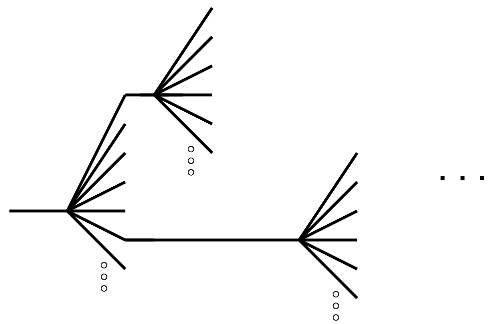
Examples

Some examples: (non-topological)

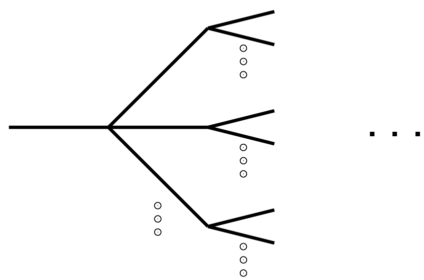
- Sacks forcing $\mathbb{S}$: all perfect trees.



- Miller forcing $\mathbb{M}$: all super-perfect trees.



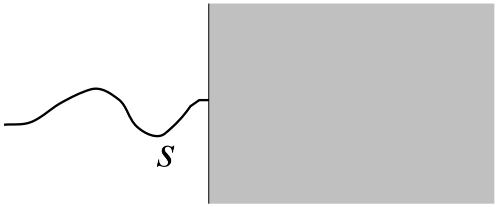
- Laver forcing $\mathbb{L}$: all trees with finite stem and afterwards ω-splitting.



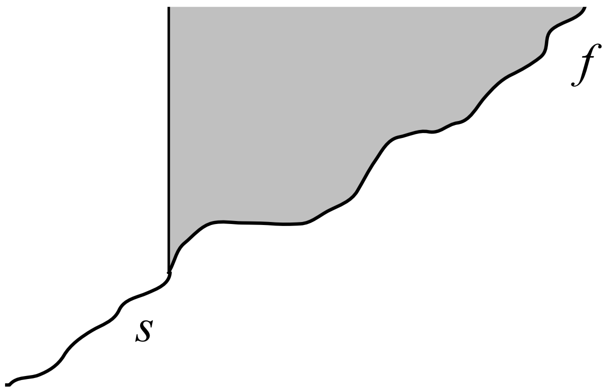
Examples (2)

Some examples: (topological)

- Cohen forcing $\mathbb{C}$: basic open sets $[s]$.



- Hechler forcing $\mathbb{D}$: for $s \in \omega^{<\omega}$ and $f \in \omega^\omega$ with $s \subseteq f$, define $[s, f] := \{x \in \omega^\omega \mid s \subseteq x \wedge \forall n \geq |s|(x(n) \geq f(n))\}$.



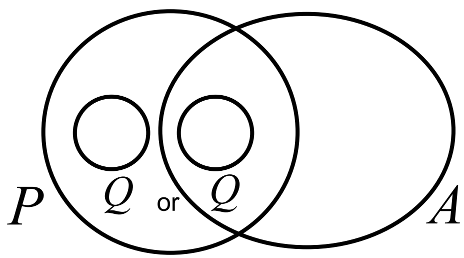
Regularity Properties

Various ways of associating regularity properties to $\mathbb{P}$.

Definition:

- For $\mathbb{P}$ non-topological: **Marczewski-Burstin** algebra:

$$A \in \mathsf{MB}(\mathbb{P}) \ :\Longleftrightarrow\ \forall P \in \mathbb{P}\ \exists Q \leq P\ ([Q] \subseteq A\ \vee\ [Q] \cap A = \varnothing)$$



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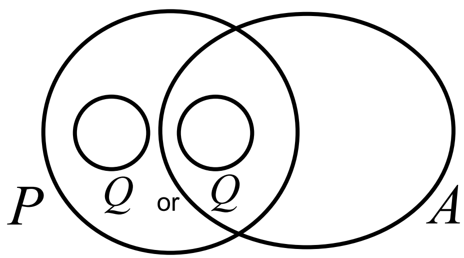

- For $\mathbb{P}$ **topological**:

$$\mathsf{BP}(\mathbb{P}) := \{A \mid A \text{ has the Baire property in } (\omega^\omega, \mathbb{P})\}$$

So far...

- Löwe considered non-topological forcings and MB$(\mathbb{P})$. Under AC, there are sets which are determined but not in MB$(\mathbb{P})$.

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- Use the following “more mathematical” characterization of determinacy:
  - A tree σ is a **strategy for Player I** if all nodes of odd length are totally splitting and all nodes of even length are non-splitting.
  - A tree τ is a **strategy for Player II** if all nodes of even length are totally splitting and all nodes of odd length are non-splitting.
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  - A set A is determined if there is a σ such that $[\sigma] \subseteq A$ or τ such that $[\tau] \cap A = \varnothing$.
- Using a Bernstein-style diagonalization procedure, find A which is deteremined but not in MB$(\mathbb{P})$.

So far...

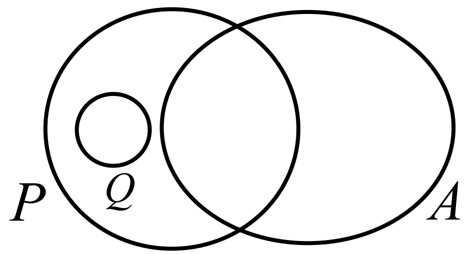
- This setting was problematic: difficulty with generalizing to “weak” version of MB, and no clear generalization for topological forcings (Baire property).
- Need new definition.

Measurability

Definition:

- $\mathbb{P}$**-nowhere-dense**:

$$A \in \mathcal{N}_{\mathbb{P}} \;:\Longleftrightarrow\; \forall P \in \mathbb{P}\; \exists Q \leq P\; ([Q] \cap A = \varnothing)$$



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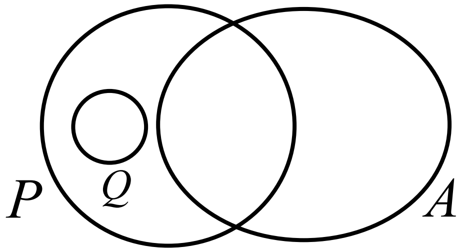

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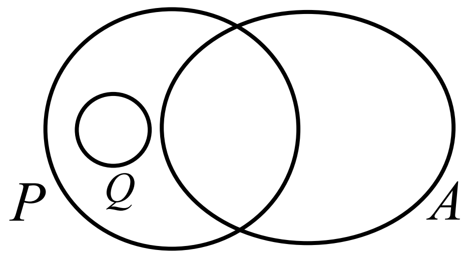
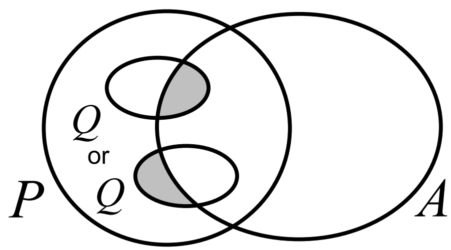


- $\mathbb{P}$**-meager**: $A \in \mathcal{I}_{\mathbb{P}}$ iff if it is a countable union of $\mathbb{P}$-nowhere-dense sets.
- Write $A \subseteq^* B$ for $A \setminus B \in \mathcal{I}_{\mathbb{P}}$. $\mathbb{P}$**-measurable**:

$$A \in \text{Meas}(\mathbb{P}) \;:\Longleftrightarrow\; \forall P \in \mathbb{P}\ \exists Q \leq P\ ([Q] \subseteq^* A \vee [Q] \subseteq^* A^c)$$



Measurability

$\mathbb{P}$-measurability is a natural generalization of the above situations.

1. If $\mathcal{I}_{\mathbb{P}} = \mathcal{N}_{\mathbb{P}}$ (fusion argument) then Meas$(\mathbb{P}) =$ MB$(\mathbb{P})$
2. If $\mathbb{P}$ is topological, then Meas$(\mathbb{P}) =$ Baire property in the $\mathbb{P}$-topology.

Both 1 and 2 can hold at the same time, e.g., Matthias forcing (Baire property in Ellentuck topology = Completely Ramsey).

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Theorem: (AC) *There is a determined set which is not in* Meas$(\mathbb{P})$.

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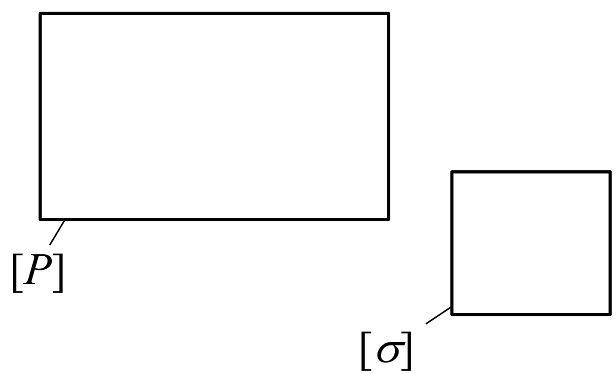

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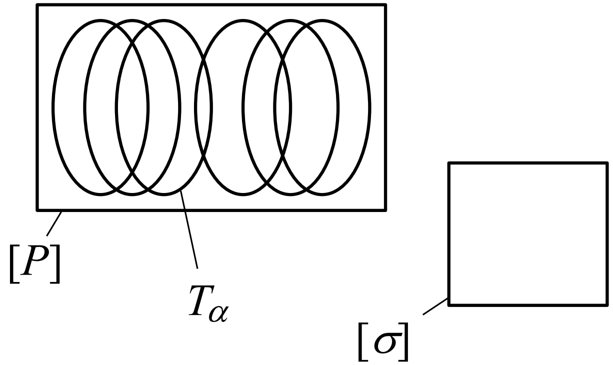

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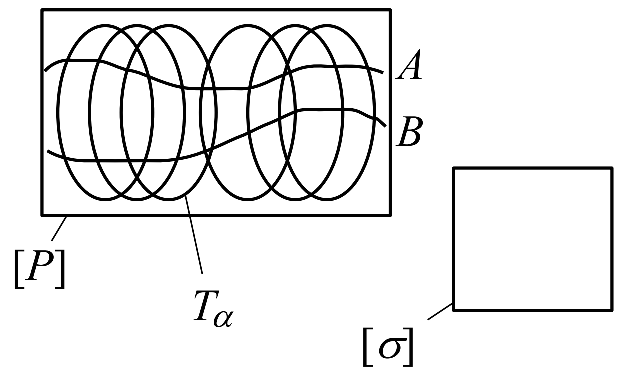
$$\forall \alpha < 2^{\aleph_0} \; (A \cap [T_\alpha] \neq \varnothing \;\wedge\; B \cap [T_\alpha] \neq \varnothing)$$



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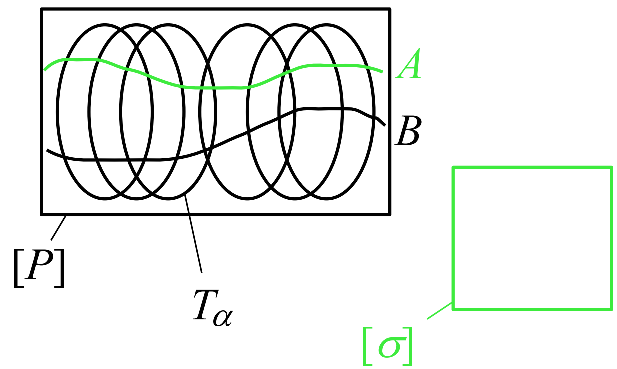
$$\forall \alpha < 2^{\aleph_0} \ (A \cap [T_\alpha] \neq \varnothing \ \wedge \ B \cap [T_\alpha] \neq \varnothing)$$
- Let $A' := A \cup [\sigma]$. Then for **no** perfect tree T in $[P]$ do we have $[T] \subseteq A'$ or $[T] \cap A' = \varnothing$, so neither A' nor its complement is in Meas$(\mathbb{P})$. But either A' or its complement is determined. $\square$

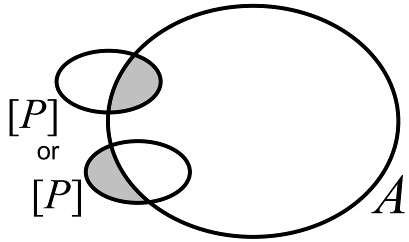


Weak Measurability

Replace measurability by a weak (local) version.

Definition: A is **weakly $\mathbb{P}$-measurable**:

$$A \in \text{wMeas}(\mathbb{P}) \;:\Longleftrightarrow\; \exists P\, ([P] \subseteq^* A \vee [P] \subseteq^* A^c)$$



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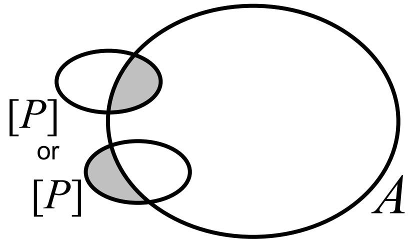

Question: does "A is determined" at least imply "A is weakly $\mathbb{P}$-measurable"?

Answer: there is a simple dichotomy.

Two Cases

- **Case 1.** For every strategy σ, there exists a $P \in \mathbb{P}$ such that $[P] \subseteq [\sigma]$.
- **Case 2.** Some strategy σ is $\mathbb{P}$-nowhere-dense.

It is not hard to see that this case distinction is exhaustive.

Main Theorem 2

Theorem:

- *In case 1,* $\mathbf{Det} \to$ *wMeas.*
- *In case 2,* $\mathbf{Det} \not\to$ *wMeas.*

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Proof:

- Case 1: trivial.
- Case 2. Fix a σ which is $\mathbb{P}$-nowhere-dense. Use this to show that for every $A \in \text{wMeas}(\mathbb{P})$ there is a perfect tree T **disjoint from** σ, s.t. $[T] \subseteq A$ or $[T] \subseteq A^c$. Now proceed similarly as before (using diagonalization). $\square$

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Examples: Sacks and Miller forcing belong to Case 1, the other standard arboreal forcings to Case 2.

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Proposition. If $\mathbb{P}$ is **not** thinner than $\mathbb{Q}$ then Meas$(\mathbb{P}) \not\subseteq$ Meas$(\mathbb{Q})$.

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