Set Theory Project: Introduction to Forcing

Assignment 2

Part A: Martin's Axiom and Generic Filters

1. Let $\mathbb{P}$ be a (forcing) partial order and $A \subseteq \mathbb{P}$. A is called a *maximal antichain* in case it is an antichain (i.e., all $p, q \in A$ are incompatible) which cannot be extended to a larger antichain (i.e., for every $p \in \mathbb{P}$ there is a $q \in A$ which is compatible to p. Show:

   (a) If $A \subseteq \mathbb{P}$ is a maximal antichain, then $\{q \in \mathbb{P} \mid \exists p \in A\, (q \leq p)\}$ is a dense subset of $\mathbb{P}$.

   (b) (AC) If $D \subseteq \mathbb{P}$ is a dense set, then there exists a maximal antichain $A \subseteq D$.

   (c) The following are equivalent for all κ:

   - if $\{D_\alpha \mid \alpha < \kappa\}$ is a collection of dense sets, then there exists a filter G, such that $G \cap D_\alpha \neq \varnothing$ for all $\alpha < \kappa$.
   - if $\{A_\alpha \mid \alpha < \kappa\}$ is a collection of maximal antichains, then there exists a filter G, such that $G \cap A_\alpha \neq \varnothing$ for all $\alpha < \kappa$.

   (Thus, in the definition of "$\mathscr{D}$-generic filter" in the statement of Martin's Axiom (and later for forcing) it does not matter whether we consider dense subsets of $\mathbb{P}$ or maximal antichains in $\mathbb{P}$.)

2. Let I be an infinite set and J an arbitrary non-empty set, and let $\mathrm{Fn}(I, J) := \{p : p \text{ is a finite function with } \mathrm{dom}(p) \subseteq I \text{ and } \mathrm{ran}(p) \subseteq J\}$. Consider the forcing $\mathbb{P} = (\mathrm{Fn}(I, J), \supseteq, \varnothing)$, i.e., $\mathbb{P}$ is the forcing with conditions from $\mathrm{Fn}(I, J)$, with the order given by $q \leq p$ iff $q \supseteq p$ (i.e., q extends p as a function), and $\mathbf{1} = \varnothing$.

   (a) Let $D_x := \{p : x \in \mathrm{dom}(p)\}$ and $R_y := \{p : y \in \mathrm{ran}(p)\}$. Show that these sets are dense, and if G is a filter which is generic for $\mathscr{D} := \{D_x : x \in I\} \cup \{R_y : y \in J\}$, then $f_G := \bigcup G$ is a surjection from I to J (i.e., it is a function, its domain is I, and its range is J).

   (b) Show that, if $|I| < |J| = \kappa$, then $\mathsf{MA}_{\mathbb{P}}(\kappa)$ is inconsistent.

Part B: Forcing basics (names and interpretations)

1. In the following, σ, τ, θ are $\mathbb{P}$-names in M and G is a $\mathbb{P}$-generic filter over M. Are the following true or false?

   (a) If $(\sigma, \mathbf{1}) \in \tau$ then $\sigma_G \in \tau_G$.

   (b) If $(\sigma, p) \in \tau$ and $p \in G$, then $\sigma_G \in \tau_G$.

   (c) If $\sigma_G \in \tau_G$ then $(\sigma, \mathbf{1}) \in \tau$.

   (d) If $x \in \tau_G$ then there exists $(\sigma, p) \in \tau$ such that $p \in G$ and $x = \sigma_G$.

   (e) If $\sigma_G \in \tau_G$ then there exists $p \in G$ such that $(\sigma, p) \in \tau$.

   (f) If $\sigma_G \in \tau_G$ then there exists $(\theta, r) \in \tau$ such that $r \in G$ and $\theta_G = \sigma_G$.

2. Let σ, τ be two $\mathbb{P}$-names in M and let G be generic over M. Show that $(\sigma \cup \tau)_G = \sigma_G \cup \tau_G$

3. Write down the $\mathbb{P}$-name $\check{3}$ in detail.

Part C: Generic extensions

1. A condition $p \in \mathbb{P}$ is called an *atom* if all q, r extending p are compatible. A forcing partial order $\mathbb{P}$ is called *atomless* if it does not contain any atoms. In Kunen, there is a proof showing that if $\mathbb{P} \in M$ is atomless than $\mathbb{P}$-generic filters over M cannot exist in M. Prove the converse, i.e., if $\mathbb{P} \in M$ is not atomless (contains at least one atom), then there exists a $G \in M$ which is a $\mathbb{P}$-generic filter over M.

2. Let M be a countable transitive model and let $\mathbb{P} \in M$ be an atomless forcing. Prove that

$$|\{G : G \text{ is a } \mathbb{P}\text{-generic filter over } M\}| = 2^{\aleph_0}.$$

3. A subset $D \subseteq \mathbb{P}$ is called *dense below* p if $\forall q \leq p \, \exists r \leq q \, (r \in D)$. Prove that if G is $\mathbb{P}$-generic over M and $p \in G$, then G has non-empty intersection with every $D \in M$ which is dense below p.