Homework Week 8
Due 17 December 2018

Recall the following definitions from topology:

Definition. A set $X \subseteq \mathbb{R}$ is **dense** if on every rational open interval (p, q) (i.e., $p < q \in \mathbb{Q}$), we have $X \cap (p, q) \neq \varnothing$. A set $X \subseteq \mathbb{R}$ is **somewhere dense** if for some (p, q), the set $X \cap (p, q)$ is dense in (p, q). A set $X \subseteq \mathbb{R}$ is **nowhere dense** if it is not somewhere dense, i.e., if

$$\forall p < q \in \mathbb{Q}\ \exists p', q' \in \mathbb{Q} \text{ s.t. } p \leq p' < q' \leq q \text{ and } (p', q') \cap X = \varnothing.$$

(alternatively, a set X is nowhere dense if its closure has empty interior, i.e., $\overline{X}^{\circ} = \varnothing$, but this is harder to use than the above definition).

By the Baire Category Theorem, the whole space $\mathbb{R}$ cannot be covered by countably many nowhere dense sets (a countable union of nowhere dense sets is called **meager** or **of first category**). Can $\mathbb{R}$ be covered by $\aleph_1$-many nowhere dense sets? Clearly, if CH holds then $\mathbb{R} = \bigcup_{r \in \mathbb{R}} \{r\}$ is an $\aleph_1$-union of singletons, each singleton being nowhere dense. The aim of this exercise is to prove the following Theorem:

Theorem 0.1. *If* $\mathsf{MA} + \neg\mathrm{CH}$ *holds, then* $\mathbb{R}$ *cannot be covered by* $\aleph_1$*-many nowhere dense sets.*

Consider the forcing $\mathbb{P}$ whose conditions are intervals $[p, q] \subseteq \mathbb{R}$, with $p, q \in \mathbb{Q} \cup \{-\infty, +\infty\}$ and $p < q$ (and $[-\infty, q]$ just means $(-\infty, q]$ etc., and we also assume $-\infty < q < +\infty$ for all $q \in \mathbb{Q}$).

The conditions are ordered by: $[p', q'] \leq_{\mathbb{P}} [p, q]$ iff $[p', q'] \subseteq [p, q]$ (iff $p \leq p' < q' \leq q$). The weakest condition $\mathbf{1}_{\mathbb{P}}$ is $(-\infty, +\infty) = \mathbb{R}$.

1. Why does $\mathbb{P}$ have the ccc? [1 point]

2. Write down what it means for $[p, q]$ and $[p', q']$ to be compatible. [1 point]

3. Let $G \subseteq \mathbb{P}$ be a filter. Show that, in general, we could have $\bigcap G = \varnothing$. [2 points]

4. Let $D := \{[p, q] : p, q \notin \{-\infty, +\infty\}\}$. Show that this set is dense and, if $G \cap D \neq \varnothing$, then $\bigcap G \neq \varnothing$. [3 points]

   [Hint: use the compactness of $[p, q]$, and the fact that finitely many conditions in G are all compatible with each other.]

5. Now, assume that $\{X_\alpha : \alpha < \kappa\}$ is a collection of nowhere dense sets. Define a collection $\mathscr{D}$ of $\mathbb{P}$-dense sets, with $|\mathscr{D}| = \kappa$, such that, if $G \subseteq \mathbb{P}$ is generic for $\mathscr{D} \cup \{D\}$ then $\bigcap G \cap X_\alpha = \varnothing$ for all $\alpha < \kappa$ (also prove that the sets you constructed are dense). Conclude the statement of the theorem. [3 points]

6. (BONUS) Define a **countable** collection $\mathscr{E}$ of $\mathbb{P}$-dense sets, such that, if G is $\mathscr{E}$-generic, then $\forall \epsilon > 0\ \exists [p, q] \in G\ (q - p < \epsilon)$. Conclude from this, that if G is $\mathscr{E}$-generic, then $\bigcap G = \{x_G\}$, for a unique point $x_G \in \mathbb{R}$. [2 bonus point]