Homework Week 7
Due 10 December 2018

1. Let $\mathbb{P}$ be a forcing-partial order and $A \subseteq \mathbb{P}$. A is called a *maximal antichain* in case it is an antichain (i.e., all $p, q \in A$ are incompatible) which cannot be extended to a larger antichain (i.e., for every $p \in \mathbb{P}$ there is a $q \in A$ which is compatible to p. Show:

   (a) If $A \subseteq \mathbb{P}$ is a maximal antichain, then $\{q \in \mathbb{P} \mid \exists p \in A\, (q \leq p)\}$ is a dense subset of $\mathbb{P}$. [1 point]

   (b) (AC) If $D \subseteq \mathbb{P}$ is a dense set, then there exists a maximal antichain $A \subseteq D$. [2 points]

   (c) The following are equivalent for all κ:

      i. if $\{D_\alpha \mid \alpha < \kappa\}$ is a collection of dense sets, then there exists a filter G, such that $G \cap D_\alpha \neq \varnothing$ for all $\alpha < \kappa$.

      ii. if $\{A_\alpha \mid \alpha < \kappa\}$ is a collection of maximal antichains, then there exists a filter G, such that $G \cap A_\alpha \neq \varnothing$ for all $\alpha < \kappa$. [2 points]

   (Thus, in the definition of "$\mathscr{D}$-generic filter" in the statement of Martin's Axiom, and later forcing, it does not matter whether we consider dense subsets of $\mathbb{P}$ or maximal antichains in $\mathbb{P}$.)

2. Let $\mathbb{C} = (\omega^{<\omega}, \supseteq, \varnothing)$ be the Cohen-forcing partial order, i.e., the forcing conditions are finite functions $p : n \to \omega$, and the ordering is given by $q \leq p$ iff $q \supseteq p$ iff $q \restriction \mathrm{dom}(p) = p$.

   (a) Let $G \subseteq \mathbb{C}$ be a filter. Show that $f_G := \bigcup G$ is a function with $\mathrm{dom}(f_G) \subseteq \omega$ and $\mathrm{ran}(f_G) \subseteq \omega$. [2 points]

   (b) For each n, let $D_n := \{p \in \mathbb{C} \ : \ n \in \mathrm{dom}(p)\}$, and $R_n := \{p \in \mathbb{C} \ : \ n \in \mathrm{ran}(p)\}$. Show that all D_n and all R_n are dense. [1 point]

   (c) Let G be a filter which is generic for the collection $\mathscr{D} = \{D_n, R_n \ : \ n < \omega\}$. Show that $f_G := \bigcup G$ is a function with $\mathrm{dom}(f_G) = \mathrm{ran}(f_G) = \omega$. [2 points]