1. Recall that a set $X \subseteq \mathbb{N}$ is called $\text{PA}$-representable if there exists a formula $\phi$ such that for all $n$ we have

$$n \in X \iff \text{PA} \vdash \phi(\dot{n})$$

$$n \notin X \iff \text{PA} \vdash \neg \phi(\dot{n})$$

Likewise, for a theory $T$ we say that a set $X$ is $T$-representable if the same holds with $T$ instead of $\text{PA}$.

(a) Show that $X := \{p : p$ is a prime number$\}$ is $\text{PA}$-representable. [1 point]

(b) Show that if $\text{PA} \subseteq T$ and $T$ is consistent, then any $X$ which is $\text{PA}$-representable is also $T$-representable. [1 point]

(c) Derive the statement “0 is the only number which is not a successor” from the axioms of $\text{PA}$. [1 point]

(d) Show that $\{\phi : \mathcal{A} \models \phi\}$ is consistent and complete for any model $\mathcal{A}$. [1 point]

2. (a) Define a binary relation $E \subseteq \omega \times \omega$ as follows: $nEm$ holds iff there is a 1 in the $n$-th place in binary expansion of $m$ (read from the right). For example, the binary expansion of 13 is 1101, therefore we have: $0E13, 1E13, 2E13, 3E13$ and $4 \notin E13$.

Show that $(\omega, E) \cong (\mathbb{V}_\omega, \in)$. [4 points]

(b) Show the relative consistency $\text{Con}(\text{PA}) \to \text{Con}(ZF \setminus \text{INF})$, without assuming any set theory in the meta-theory (note that if ZFC is the meta-theory then the assertion is trivial, since $\mathbb{V}_\omega \models ZF \setminus \text{INF}$). This shows that $ZF \setminus \text{INF}$ is logically not any stronger than $\text{PA}$. [2 points]