Homework Week 6
Due 3 December 2018

1. Recall that a set $X \subseteq \mathbb{N}$ is called PA-*representable* if there exists a formula ϕ such that for all n we have

$$n \in X \;\Leftrightarrow\; \mathsf{PA} \vdash \phi(\dot{n})$$
$$n \notin X \;\Leftrightarrow\; \mathsf{PA} \vdash \neg\phi(\dot{n})$$

Likewise, for a theory T we say that a set X is T-representable if the same holds with T instead of PA.

(a) Show that $X := \{p \,:\, p \text{ is a prime number}\}$ is PA-representable. [1 point]

(b) Show that if $\mathsf{PA} \subseteq T$ and T is consistent, then any X which is PA-representable is also T-representable. [1 point]

(c) Derive the statement "0 is the only number which is not a successor" from the axioms of PA. [1 points]

(d) Show that $\{\phi \,:\, \mathcal{A} \models \phi\}$ is consistent and complete for any model $\mathcal{A}$. [1 point]

2. (a) Define a binary relation $E \subseteq \omega \times \omega$ as follows: nEm holds iff there is a 1 in the n-th place in binary expansion of m (read from the right). For example, the binary expansion of 13 is 1101, therefore we have: $0E13$, $1\not{E}13$, $2E13$, $3E13$ and $4\not{E}13$. Show that $(\omega, E) \cong (V_\omega, \in)$. [4 points]

(b) Show the relative consistency $\mathrm{Con}(\mathsf{PA}) \to \mathrm{Con}(\mathsf{ZF} \setminus \textsc{Inf})$, *without* assuming any set theory in the meta-theory (note that if ZFC is the meta-theory then the assertion is trivial, since $V_\omega \models \mathsf{ZF} \setminus \textsc{Inf}$). This shows that $\mathsf{ZF} \setminus \textsc{Inf}$ is logically not any stronger than PA. [2 points]