Homework Week 4
Due 12 November 2018

1. (a) Let M be a proper class model of ZFC. Show that for every set x, if $x \subseteq M$ then there exists a set $y \in M$ such that $x \subseteq y$. [2 points]

   Hint: Think of V_α^M

   (b) Let $M \neq \varnothing$ be any class and assume that for every set x, if $x \subseteq M$ then $x \in M$. Show that then $M = V$. [2 points]

2. Prove the following:

   (a) Let M be an *elementary submodel* of N, i.e., $(M, \in) \preccurlyeq (N, \in)$. Let $c \in N$ be an element which is *uniquely definable in* N; that means that there exists a formula $\phi(x)$ such that
   $$N \models \forall x\, (\phi(x) \leftrightarrow x = c).$$
   Then $c \in M$. [3 points]

   (b) If $M \preccurlyeq H_{\omega_2}$ then $\omega_1 \in M$. [1 point]

   (c) If $M \preccurlyeq V_\omega$ then $M = V_\omega$. [2 points]

   Hint: Prove, by $\in$-induction, that every $x \in V_\omega$ is uniquely definable in V_ω (in the sense of (a)).

3. (Bonus) Find the mistake in the argument below: [2 bonus points]

> **Theorem.** ZFC *is inconsistent.*
>
> *Proof.* The Reflection theorem says: for every finite $F \subseteq \mathsf{ZFC}$, there is a *set-model* $M \models F$. In other words, every finite $F \subseteq \mathsf{ZFC}$ is *satisfiable.* But the compactness theorem of first order logic says: if every finite subset of a theory is satisfiable, then the theory itself is satisfiable. Therefore ZFC is satisfiable. All of this is provable in ZFC, hence $\mathsf{ZFC} \vdash \mathrm{Con}(\mathsf{ZFC})$. But by Gödel's Incompleteness Theorem, this implies that ZFC is inconsistent. □