Homework Week 3
Due on 5 November 2018

1. Prove that if κ is regular, then:
   (a) For all $x, y \in H_\kappa$ we have: $|x| = |y|$ if and only if $H_\kappa \models |x| = |y|$. [2 points]
   (b) For $\lambda < \kappa$ we have: λ is a cardinal if and only if $H_\kappa \models (\lambda$ is a cardinal$)$. [1 point]
   (c) $H_{\kappa^+} \models \forall x\, (|x| \leq \kappa)$. In particular, $H_{\omega_1} \models$ every set is countable.
   (Therefore $\mathrm{Con}(\mathsf{ZFC}) \to \mathrm{Con}(\mathsf{ZF} \setminus \mathsf{Pow} +$ "every set is countable"$))$. [1 point]

2. Show that $V_{\omega+\omega}$ does not satisfy the statement "every well-order is isomorphic to an ordinal". Track the exact instance of Replacement which fails in $V_{\omega+\omega}$. [2 + 1 points]

   Hint: define a well-order on $\omega \times 2$.

3. Let κ be strongly inaccessible.
   (a) Show that the following statements are absolute for $H_\kappa = V_\kappa$, for $\lambda < \kappa$: [2 points]
   - λ is regular.
   - λ ist strongly inaccessible.

   (b) Use the above to provide an alternative proof of

   $$\mathrm{Con}(\mathsf{ZFC}) \to \mathrm{Con}(\mathsf{ZFC} + \text{"there are no inaccessible cardinals"})$$

   which does not use Gödel's Incompleteness Theorem. [1 point]

   Hint: Consider the *least* strongly inaccessible cardinal, if it exists.