Homework Week 2
Due on 29 October 2018

1. Find the mistake in the argument below. Then, explain (in your own words) what the difference is between this argument and the (correct) proof of Tarski's theorem on the undefinability of the truth predicate. [2 points]

> **Theorem.** ZFC *is inconsistent.*
>
> *Proof.* Let $\{\theta_n : n < \omega\}$ be an enumeration of all formulas of $\mathcal{L}_\in$ with exactly one free variable. Let $\psi(x)$ be the formula "$x \in \omega \wedge \neg\theta_x(x)$". Since ψ is a formula of $\mathcal{L}_\in$ in one free variable, there exists $e \in \omega$ such that $\psi = \theta_e$. But then $\mathsf{ZFC} \vdash \theta_e(e) \leftrightarrow \psi(e) \leftrightarrow \neg\theta_e(e)$. □

2. A formula is called Σ_1 if it has the form $\exists x_0 \dots \exists x_k\, \theta$ for a Δ_0-formula θ, and Π_1 if it has the form $\forall x_0 \dots \forall x_k\, \theta$ for a Δ_0-formula θ.

   (a) Let M be a transitive model. Show that for all Σ_1-formulas ϕ we have: $\phi^M \to \phi$, und for all Π_1-formulas ψ we have: $\psi \to \psi^M$ (we call the former *upwards absoluteness* and the latter *downwards absoluteness*). [1 point]

   (b) In the lecture, we remarked that "being a cardinal", "being of the same cardinality" and similar statements are, in general, not absolute.

   Prove that the statement "$|x| = |y|$" is upwards absolute for transitive models, and the statement "κ is a cardinal" is downwards absolute for transitive models. (You may use the fact that "f is a function", "f is a bijection", "α is an ordinal", and the concepts $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ are all Δ_0 and therefore absolute). [2 point]

3. In the lecture we defined the relativization ϕ^M by changing the "domain" to the class model M, but not touching the $\in$-relation. Now let M be a class and $E \subseteq M \times M$ a class relation.

   (a) Give a definition of the relativization $\phi^{(M,E)}$, which formalizes the idea that ϕ holds in the model (M, E), i.e., the model with domain M and the relation-symbol $\in$ interpreted as E. [1 point]

   (b) Let $F : V \to V$ be a bijective class-function. Define $E \subseteq V \times V$ by:

   $$xEy \;:\Leftrightarrow\; x \in F(y).$$

   We claim that (V, E) is a model of $\mathsf{ZFC} - \mathsf{Foundation}$. Choose any two axioms of $\mathsf{ZFC} - \mathsf{Foundation}$, and prove that they hold in (V, E). [2 points]

   (c) Use the previous claim to show

   $$\mathrm{Con}(\mathsf{ZFC}) \;\to\; \mathrm{Con}(\mathsf{ZFC} - \mathsf{Foundation} + \text{“}\exists x\,(x = \{x\})\text{”})$$

   [Hint: use $F(0) := 1, F(1) := 0$ and $F(x) = x$ for all other x]. [2 points]