Practice exercises for Monday 10 December

1. Let I, J be any sets and let $\mathrm{Fn}(I, J) := \{p : p \text{ is a finite function with } \mathrm{dom}(p) \subseteq I \text{ and } \mathrm{ran}(p) \subseteq J\}$. Consider the forcing $\mathbb{P} = (\mathrm{Fn}(I, J), \supseteq, \varnothing)$, i.e., $\mathbb{P}$ is the forcing with conditions from $\mathrm{Fn}(I, J)$, with the order given by $q \leq p$ iff $q \supseteq p$ (i.e., q extends p as a function), and $\mathbf{1} = \varnothing$. Show:

   (a) Let $D_x := \{p : x \in \mathrm{dom}(p)\}$ and $R_y := \{p : y \in \mathrm{ran}(p)\}$. Show that these sets are dense, and if G is a filter which is generic for $\mathcal{D} := \{D_x, R_y : x \in I, y \in J\}$, then $f_G := \bigcup G$ is a surjection from I to J.

   (b) Show that, if $|I| < |J| = \kappa$, then $\mathsf{MA}_{\mathbb{P}}(\kappa)$ is inconsistent.

2. (A bit tricky!) Show that it is **not possible** to define a proper class $\boldsymbol{M}$ and prove (from ZFC) that $\boldsymbol{M}$ is a (class) model of $\mathsf{ZFC} + \neg\mathrm{CH}$. This means, it is not possible to find a formula ϕ such that $\boldsymbol{M} = \{x : \phi(x)\}$, and prove that for every ZFC-axiom θ, the relativized version $\theta^{\boldsymbol{M}}$ holds and also $(\neg\mathrm{CH})^{\boldsymbol{M}}$ holds.

   Then how are we supposed to show $\mathrm{Con}(\mathsf{ZFC}) \to \mathrm{Con}(\mathsf{ZFC} + \neg\mathrm{CH})$?