Practice exercises for Monday 5 November

1. Let M be any transitive model of ZFC, or "sufficiently much of ZFC" (don't worry exactly how much). Prove that:

   (a) If $x \subseteq M$ is a finite set, then $x \in M$.

   (b) $V_\omega \subseteq M$ (in particular, all finite ordinals are in M).

   (c) If $f := \langle x_0, \dots, x_k \rangle$ is a finite sequence with $x_i \in M$ for all i, then $f \in M$.

2. Let M be a *countable* transitive model of ZFC, or "sufficiently much of ZFC". Show that $o(M) := \mathrm{Ord} \cap M$ is a countable ordinal $> \omega$.

3. Define $\beth_0 := \aleph_0$, $\beth_{\alpha+1} := 2^{\beth_\alpha}$ and $\beth_\lambda = \bigcup_{\alpha<\lambda} \beth_\alpha$ for limits λ.

   (a) Show that $|V_{\omega+\alpha}| = \beth_\alpha$ for all α, and that the Generalized Continuum Hypothesis GCH is the statement
   $$\forall \alpha \; (\aleph_\alpha = \beth_\alpha).$$

   (b) Show that $|H_\kappa| = 2^{<\kappa}$ for all $\kappa > \omega$.

   (c) Show that $\kappa > \omega$ is inaccessible if and only if $\beth_\kappa = \kappa$.