Practice exercises for Monday 29 October

1. Prove the following in detail: if ϕ and ψ are formulas such that $\mathsf{ZFC}^* \vdash \phi \leftrightarrow \psi$ and ψ is Δ_0, then ϕ is absolute for transitive models M such that $M \models \mathsf{ZFC}^*$. Here ZFC^* denotes some sufficiently large fragment of ZFC.

   Note: this means that when you show that ϕ is "equivalent" to a Δ_0-formula, you need to be careful how this equivalence is proved! If M does not satisfy sufficiently much of ZFC to prove this "equivalence", then ϕ might fail to be absolute for M even though it is (in ZFC) equivalent to a Δ_0-formula.

2. We want to show that the statement: "the relation R on A is well-founded" is absolute for transitive models M of ZFC^* (where ZFC^* is anything sufficient to prove Lemma 1 below.)

   (a) **Lemma 1.** *(A, R) is well-founded iff there exists a* rank function*, i.e., a mapping* $\mathrm{rk} : A \to \mathrm{Ord}$ *such that* $aRb \to \mathrm{rk}(a) < \mathrm{rk}(b)$.

   (b) Show that "(A, R) is well-founded" is a Π_1-statement.

   (c) Show that "(A, R) is well-founded" is ZFC^*-equivalent to a Σ_1-statement.

   (d) Formulas which are both (equivalent to) Σ_1 and Π_1 and called Δ_1. Since they are both downwards- and upwards-absolute, they are absolute for transitive models (which can prove this equivalent).