**Practice exercises for Monday 14 January**

1. (a) Suppose that $\forall p \in \mathbb{P}\ \exists q \leq p\ (q \Vdash \phi)$. Then $\mathbf{1} \Vdash \phi$.
   (b) Similarly, suppose $p_0 \in \mathbb{P}$ is such that $\forall p \leq p_0\ \exists q \leq p\ (q \Vdash \phi)$. Then $p_0 \Vdash \phi$.
   (c) The following are equivalent:
      i. $p_0 \Vdash \phi$,
      ii. $\forall p \leq p_0\ (p \Vdash \phi)$, and
      iii. $\{p \leq p_0\ :\ p \Vdash \phi\}$ is dense below $p_0$.

2. (a) For $p \in \mathbb{P}$ and $\phi$ in the forcing language, we say $p$ *decides* $\phi$ if $p \Vdash \phi$ or $p \Vdash \neg\phi$. Show that for every $p \in \mathbb{P}$ and every $\phi$, there is $q \leq p$ which decides $\phi$.
   (b) Let $\tau$ be a name such that $p \Vdash \tau \in \check{\omega}$. Show that there exists $q \leq p$ and $n \in \omega$ such that $q \Vdash \tau = \check{n}$. We say that $q$ *decides* $\tau$.
   (c)* A forcing partial order $\mathbb{P}$ is called *σ-closed*, if for any decreasing sequence $p_0 \geq p_1 \geq p_2 \geq \dots$ there exists a condition $q \in \mathbb{P}$ such that $p_n \geq q$ for all $n$ (not all forcings are $\sigma$-closed, in fact, all the ones that we have in our lecture are *not* $\sigma$-closed).

   Let $\dot{f}$ be a $\mathbb{P}$-name such that $p_0 \Vdash \dot{f} : \omega \to \omega$. Prove that there exists a function $g : \omega \to \omega$ in $M$ and a $q \leq p$ such that $q \Vdash \dot{f} = \check{g}$.

   Conclude form this that if $\mathbb{P}$ is a $\sigma$-closed forcing and $G$ is $\mathbb{P}$-generic over $M$, then $\omega^\omega \cap M = \omega^\omega \cap M[G]$ (i.e., $\mathbb{P}$ does not add new functions from $\omega$ to $\omega$).