Non well-founded set theory

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1 Introduction

In what follows, ZFC^- will denote ZFC minus the Foundation axiom. The main purpose of this presentation is to prove the consistency of an extension of this theory, involving a new axiom that we shall discuss later. I will follow very closely [3]. I will start by shortly explain some of the motivations behind the conception of such a theory (Introduction). I will quickly move on to introduce some key concepts and results that will be needed for the announced proof of consistency (Sections 2), and will conclude with this proof (Section 4).

When paradoxes arised at the begining of the XX$^{\text{th}}$ century, circularity was soon denounced as closely connected to them[1]. A natural reaction was thus to avoid any kind of circularity by technical means, for example by typing the universe of discourse or by restraining the naive schema of comprehension. A side effect of these treatments was that the universe in which mathematics were to be formalised – the set theoretical universe – was restrained as well : only the so-called well-founded sets were allowed to exist.

Beyond the paradoxes, at least two other factors contributed to exclude non-well-founded sets. On the one hand, the axiom of foundation is needed to prove that every set appears at some stage of the Von Neumann hierarchy, a result which is both very elegant and convenient. On the other hand, one could argue that well-founded sets are intuitively prior, or at least easier to manipulate and understand for a start[2]. Before the end of the XX$^{\text{th}}$ century, it was difficult to make sense of collections which *actualy* never end. One revolutionary (and thus controversial) aspect of Cantor's work was precisely to consider actual infinite collections. However, though Cantor's sets may never end, they must begin somewhere. After Cantor's inovations towards a science of infinity, a step further might be to extend the universe of sets by welcoming the non-well-founded ones.

Today, we could say that though natural, set-theoretists' reaction regarding circularity was undoubtedly too radical. There are more subtil ways of getting rid of a scratch on

[1]See the discussion between B. Russell and H. Poincaré in *Revue de métaphysique et de morale*, Vol. 14, 1906, in particular [5], [6]

[2]On this point and its connection to the iterative conception of sets, see the introduction of [1].

the leg than cutting it off entirely. Indeed, if manipulated not carefuly, circularity *can* lead to contradictions. But not every circular object *does* lead to contradictions. And in fact, there are circular objects that do diserve to be studied from a mathematical point of vue, and thus represented in a suitable set theory[3].

2 A graphical representation of sets

In this section, we introduce some key concepts and results of graph theory.

Definition 1. *A graph $\mathcal{G}$ is given by an oredred pair (G, R) where G is a non-empty set of nodes together with a binary relation R on G. The elements of R are called edges. Whenever $(x, y) \in R$, we say that x is a parent of y and that y is a child of x. A path in G is a sequence of nodes, each of which (except the firt one) is a child of its predecessor. A graph is well-founded if it has no infinite path. For all $x \in G$, $ch_G(x)$ will denote the set of all children of x in G. A graph is rooted if it contains a unique node n_0 (the "top node") such that every path starts at n_0.*

Example : $\mathcal{G}_1 = (N, \rightarrow)$ where $N = \{n_0, n_1, n_2, n_3\}, \rightarrow = \{(n_3, n_2), (n_3, n_1), (n_3, n_0), (n_2, n_1), (n_2, n_0), (n_1, n_0)\}$ and $\mathcal{G}_2 = (\{0\}, =)$ are graphs. In $\mathcal{G}_1$, n_3 is a parent of n_2. $n_2 \rightarrow n_1 \rightarrow n_0$ and $0 = 0 = 0 = ...$ are respectively paths in $\mathcal{G}_1$ and $\mathcal{G}_2$. $\mathcal{G}_2$ is well-founded, but $\mathcal{G}_3$ is not. $ch_{G_1}(n_2) = \{n_1, n_0\}$ and $ch_{G_2}(0) = \{0\}$. $\mathcal{G}_1$ is rooted, $\mathcal{G}_2$ is not.

Note that all the graphs we will be working with are rooted. Hence from now on whenever I write "graph", read "rooted graph". The two fundamental operations we will perform on graphs will consist of *tagging* and *decorating* them.

Definition 2. *Let $\mathcal{G}$ be a graph with top node n_0. A tagging of $\mathcal{G}$ is a map from the set of childless nodes to $\{\emptyset\}$.*

Remark. If we were to work with atoms, a tagging of $\mathcal{G}$ would be a map from the set of childless nodes to $\{\emptyset\} \cup \mathcal{A}$ where $\mathcal{A}$ is a class of atoms. This remark extends to the following definition.

Definition 3. *Let $\mathcal{G} = (G, R)$ be a graph and t be a tagging of $\mathcal{G}$. A decoration of $\mathcal{G}$ is a map d such that $d(x) = t(x)$ if $x \in dom(t)$, and $d(x) = \{d(y) : (x, y) \in R\}$.*

Given a set x, any tagged graph with a decoration which assigns x to the top node is called a *picture* of x. It is easy to verify that every set has at least one picture. Now one can formulate the Foundation Axiom in these terms : *no non-well-founded graph can be decorated.* Intuitively, it is easy to see why. By Foundation, there is no non-well-founded

[3]For a discussion on these objects, see [2] or [4]

set, therefore there is no function that assigns a set to a non-well-founded graph. On the contrary, the Anti-Foundation Axiom (AFA) states that *every tagged graph has a unique decoration.* In particular, a non-well-founded graph *can* be decorated, *i.e* there exists a (unique) set that can be assigned to its top node.

Remark. In ZFC, the axiom of extensionality guarantees that the identity of a set is entirely determined by its elements. In this perspective, extensionality also provides an informative (non-trivial) question that one should ask himself when wondering if two sets are equal : are their elements equal ? Given $a = \{c\}$ and $b = \{d\}$, the problem "Is a equal to b ?" reduces to "Is c equal to d ?"[4]. However, this informative potential is lost when applied to non-well-founded sets : given $a = \{b\}$ and $b = \{a\}$, extensionality tells us that $a = b$ if and only if $a = b$... Several anti-foundations axioms were proposed depending on the way one wants to treat equality of sets when dealing with non-well-founded ones. One of the approach, the first one according to Aczel, during the 60's and the 70's by Maurice Boffa, was simply to stick to extensionality as the sole axiom governing identity.

Proving that $ZFC^- + AFA$ is consistent is proving that it has a model. Such a model will be built out of a generalization of the concept of a graph, namely a system.

Definition 4. *A system is an ordered pair $\mathcal{S} = (S, R)$ where S is a class of nodes, R is a class of edges satisfying the requirement that for any node s in S, $ch_S(s)$ must be a set.*

Any graph is a system, but the converse does not hold : $(V, \longrightarrow)$ with $x \longrightarrow y$ if $y \in x$ is a system but not a graph. Taggings and decorations can be extended in a natural way to systems. Assuming AFA (every tagged graph has a unique decoration), one can prove that every labeled tagged system has a unique decoration.

3 Relative consistency of $ZFC^- + AFA$

Let M be a system. We will say that a relation R on M is a *bisimulation* on M if for all $x, y \in M$, whenever $(x, y) \in R$ then any child of x must be R-related to some child of y and *vice-versa.*

Example : for two sets x, y, write $x \equiv y$ if and only if there is a graph M that pictures both x and y. Then $\equiv$ is a binary relation on the system $\mathcal{V} = (V, \longrightarrow)$ and one can prove that it is a bismulation on $\mathcal{V}$. A bisimulation on M will be called *small* if it is a set. Define $\equiv_M$ on M by $x \equiv_M y$ if and only if $(x, y) \in R$ for some small bisimulation on M.

Lemma 1. *Let M be a system. Then the relation $\equiv_M$ is the unique maximal bisimulation on M. That is : (i) $\equiv_M$ is a bisimulation on M and (ii) if R is any bisimulation on M, then for any $x, y \in M$, if Rxy holds then $x \equiv_M y$.*

[4] Besides, note that by foundation this process will always stop, even though it might stop after infinitely many steps.

Theorem 1. *For all $x, y \in V$, let $x \equiv y$ if and only if there is a graph G that both pictures x and y. Then $\equiv$ is the maximal bisimulation on V. Formaly, for all sets x, y, $x \equiv y$ if and only if $x \equiv_V y$.*

Definition 5. *A system M is said to be extensional if $x \equiv_M y$ implies $x = y$ for all $x, y \in M$.*

Theorem 2. *The following are equivalent : (i) Every graph has at most one decoration ; (ii) The system $(V, \rightarrow)$ is extensional.*

Remark. "Every graph has at least one decoration" is equivalent to "every extensional system is an exact picture".

Definition 6. *A system map from M to M' is a map $f : M \rightarrow M'$ such that for all $x \in M$, $ch_{M'}(f(x)) = \{f(y) : y \in ch_M(x)\}$.*

In words, $f : M \rightarrow M'$ is a system map if for all $x \in M$, f maps the children of x onto the children of $f(x)$.

Definition 7. *Let R be a bisimulation equivalence relation on some system M. We say that M' is the quotient of M by R if and only if there is a surjective map $f : M \rightarrow M'$ such that for all $x, y \in M$, xRy holds if and only if $f(x) = f(y)$ holds as well.*

In short, a quotient of M by R is simply a system M' in which all the nodes linked by R are identified. Surjectivity tells us that M' is nothing more than that. As will become clear later, our main interest in quotients here concerns the extensional ones. Note that one can prove[5] that a quotient of M by R is extensional if and only R is $\equiv_M$. Using this result, it is also possible to prove that any system has an extensional quotient (the proof uses AC).

Theorem 3. *Let M be any system. Then the following are equivalent : (i) M is extensional ; (ii) for each system M', every system map $f : M \rightarrow M'$ is one-one ; (iii) for each small system M_0 there is at most one system map $f : M_0 \rightarrow M$*

Given a system M, an *M-decoration* of a graph G is a system map $f : G \rightarrow M$. In particular, a V-decoration of G is simply a decoration of G. A system is called *M-complete* if every graph has a unique M-decoration (AFA says that V is a complete system).

Let $V_0 = \{G_x, G_y, ...\}$ be the class of all graphs, where $x, y...$ are the top nodes of the graph they index. We define the system (V_0, R) such that R contains (G_x, G_y) if and only if $(x, y) \in R_G$ (R_G being the set of edges in G). Let $f_q : V_0 \rightarrow V_q$ be the extensional quotient of V_0 (I mentioned earlier that any system has an extensional quotient).

[5] [3], Lemma 7.8.8.

Lemma 2. *Let M be any system. Then there is a unique system map $f : M \to V_q$.*

As a corolary, V_q is complete : every graph has a unique V_q-decoration. Given any system M, we may obtain an interpretation of the language of set theory by letting the variables range over the nodes of M and interpreting the predicate symbol $\in$ by the relation $\in_M$ with the following semantics : $x \in_M y$ if and only if $y \to x \in R$ for all $x, y \in M$.

We are almost there : we are about to show that V_q is a model of $ZFC^- + AFA$. The steps will run as follows. First we will introduce a new concept, namely a "full" system. After noticing that any complete system is full, we will prove that any full system is a model of ZFC^-, which will establish that any complete system is a model of ZFC^-. Then we will show that any complete system is also a model of AFA. Since V_q is complete, it will follow that V_q is a model of $ZFC^- + AFA$.

A system is *full* if for every set $u \subseteq M$, there is a unique element $x \in M$ such that $u = ch_M(x)$. For example, V is a full system. One can prove that every complete system is full.

Theorem 4. *Every full system is a model of ZFC^-.*

Let M be a full system. For all $u \subseteq M$, we write u^M to denote the unique a such that $u = ch_M(a)$. We verify that each axiom of ZFC^- is true in $(M, \in_M)$:

- *Extensionality.* Suppose $M \models \forall x(x \in m \leftrightarrow x \in n)$ for some $m, n \in M$. Then $ch_M(m) = ch_M(n)$. But $m = ch_M(m)^M$ and $n = ch_M(n)^M$. Hence $M \models m = n$.

- *Pairing.* Let $m, n \in M$. Then $\{m, n\} \subseteq M$, so let $c = \{m, n\}^M$. Then $ch_M(c) = \{a, b\}$, so $M \models m \in c \land n \in c$.

- *Union.* Let $m \in M$. Then $x = \bigcup\{ch_M(y) : y \in ch_M(m)\}$ is a subset of M, so let $c = x^M$. Then $M \models \forall y \forall z((y \in m \land z \in y) \to z \in c)$.

- *Power set.* Let $a \in M$ and define $x = \{y^M : y \subset ch_M(a)\}$. Then $x \subseteq M$, so let $c = x^M$. Then $M \models \forall v(\forall z(z \in v \to z \in a) \to v \in c)$.

- *Infinity.* Let $\theta_0 = \emptyset^M$, $\theta_{n+1} = (ch_M(\theta_n) \cup \{\theta_n\})^M$ with $n \in \mathbb{N}$. Then $\theta_n \in M$ for all $n \in \mathbb{N}$, so let $\theta = \{\theta_n : n \in \mathbb{N}\}^M \in M$. Clearly, $M \models [\theta_0 \in \theta \land (\forall x \in \theta)(\exists y \in \theta)(x \in y)]$.

- *Separation.* Let $a \in M$ and $\varphi(x)$ be a formula with at most x free written on the signature $\{\in\}$ and eventualy containing constants from M. Define $c = \{b \in ch_M(a) : M \models \varphi(b)\}^M$. Then $M \models \forall x(x \in c \leftrightarrow x \in a \wedge \varphi(a))$.

- *Collection.* Let $a \in M$ and $\varphi(x, y)$ be a formula with at most x, y free written on the signature $\{\in\}$ and eventualy containing constants from M. Suppose that $M \models (\forall x \in a)\exists y\varphi(x, y)$. Then $(\forall x \in ch_M(a))(\exists y \in B)(y \in M \wedge M \models \varphi(x, y))$. By the collection schema, there is a set b such that $\forall x \in ch_M(a)(\exists y \in b)(y \in M \wedge M \models \varphi(x, y))$. Let $c = (b \cap M)^M$. Then $M \models (\forall x \in a)(\exists y \in c)\varphi(x, y)$.

- *Choice.* Let $a \in M$ be such that $M \models (\forall x \in a)(\exists y)(y \in x)$ (a is a set of non-empty sets) and $M \models (\forall x_1, x_2 \in a)(\exists y(y \in x_1 \wedge y \in x_2) \rightarrow x_1 = x_2)$ (the elements of a are pairwise disjoint). Thus $\{ch_M(x) : x \in ch_M(a)\}$ is a set of non-empty pairwise disjoint sets. By AC, there exists b such that for each $x \in ch_M(a)$, the set $b \cap ch_M(x)$ has a unique element $c_x \in M$. Then $c = \{c_x : x \in ch_M(a)\}^M$ is such that $M \models (\forall x \in a)(\exists y \in x)(\forall u \in x)(u \in c \leftrightarrow u = y)$

As we saw earlier, every complete system is full. Since we just proved that every full system is a model of ZFC^-, we can conclude that every complete system is a model of ZFC^-. In particular, V_q is a model of ZFC^-. All that is left to do concerns AFA.

Theorem 5. *Every complete system is a model of AFA.*

Proof. Let M a complete system. For $a, b \in M$, define the M-ordered pair $(a, b)_M$ by $(a, b)_M = \{\{a\}^M, \{a, b\}^M\}^M$. For $c \in M$, define $M \models c$ is a graph if and only if there are $a, b \in M$ such that $c = (a, b)_M$ and it is true in M that b is a binary relation on a. This last requirement reduces to $ch_M(b) \subseteq \{(x, y)_M : x, y \in ch_M(a)\}$.

Hence, if $c \in M$ is such that $M \models c$ is a graph, we may define a genuine graph G by taking a, b as above and letting the elements of $ch_M(a)$ be the nodes of G and the ordered pairs (x, y) such that $(x, y)_M \in ch_M(b)$ the edges. Since M is complete, G has a unique M-decoration, let it be d. Then $d : ch_M(a) \rightarrow M$, and for all $x \in ch_M(a)$, $d(x) = \{d(y) : (x, y)_M \in ch_M(b)\}$. Set $f = \{(x, d(x))_M : x \in ch_M(a)\}^M$. Then $f \in M$, and it is routine to verify that $M \models f$ is the unique decoration of the graph c.

In fact, by virtue of lemma 2, there is a unique system map $f : V \rightarrow V_q$, so V_q is a model of $ZFC^- + AFA$ that canonicaly embeds V. Thus we may regard our construction of V_q as an extension of V[6].

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References

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