Advanced Set Theory: Infinite Games and Determinacy

Assignment 3

Exercise 1. (Unfolding)

The aim of this exercise is to see that determinacy in a point-class $\boldsymbol{\Gamma}$ can sometimes lead to regularity properties for the wider class of **projections** of sets in $\boldsymbol{\Gamma}$.

Definition. Let $\boldsymbol{\Gamma}$ be a boldface pointclass. The *projection class of* $\boldsymbol{\Gamma}$ is

$$\boldsymbol{\Gamma}^{\exists} := \{A \subseteq \omega^\omega \ : \ A = p[C] \text{ for some } C \in \boldsymbol{\Gamma}\}$$

where $C \subseteq \omega^\omega \times \omega^\omega$ and $p[C] = \{x : \exists y\, ((x, y) \in C)\}$. A typical example are the classes in the projective hierarchy, i.e., $\boldsymbol{\Sigma}^1_{n+1} = \left(\boldsymbol{\Pi}^1_n\right)^{\exists}$.

Consider the following game $G^{**}_{\exists}(C)$:

$$\begin{array}{r|cccccc} \text{I}: & (s_0, y(0)) & & (s_1, y(1)) & & \dots & \\ \hline \text{II}: & & t_0 & & t_1 & & \dots \end{array}$$

so that

- $s_i, t_i \in \omega^{<\omega} \setminus \{\varnothing\}$.
- $y(i) \in \omega$.

Letting $x := s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$ and $y = \langle y(0), y(1), y(2), \dots \rangle$, Player I wins the game iff $(x, y) \in C$.

Intuitively, this game is just like the Banach-Mazur game $G^{**}(A)$ which we used to prove the Baire property, except that Player I additionally plays a single digit $y(n)$ at every move, which will eventually result in the "witness" y for the fact that $x \in p[C]$.

Use the above game to prove the following theorem:

Theorem. *Assume every set in* $\boldsymbol{\Gamma}$ *is determined. Then every set in* $\boldsymbol{\Gamma}^{\exists}$ *satisfies the Baire property.*

Remarks:

1. You may assume, without proof, that the above game can be re-cast as a game with natural numbers, i.e., that if $C \in \boldsymbol{\Gamma}$ then $G^{**}_{\exists}(C)$ is determined.
2. This provides an alternative ZFC-proof that analytic ($\boldsymbol{\Sigma}^1_1$) sets satisfy the Baire property (this was first proved in 1917 by Suslin using classical methods).
3. Exercise 27.14 from Kamanori contains the structure of the argument, so the advice is to first try to see how far you get on your own, and then consult Kanamori.

Exercise 2. (Wadge reducibility)

1. Show that for any set $A \notin \{\varnothing, \omega^\omega\}$, we have both $\varnothing <_W A$ and $\omega^\omega <_W A$.
2. Show that $\varnothing \not\leq_W \omega^\omega$ and $\omega^\omega \not\leq_W \varnothing$. Conclude that $[\varnothing]_W = \{\varnothing\}$ and $[\omega^\omega]_W = \{\omega^\omega\}$.
3. If $(P, \leq)$ is a partial order, then a subset $A \subseteq P$ is called an *antichain* if $\forall p, q \in A$ ($p \not\leq q \wedge q \not\leq p$). Show that in the partial order $(\mathbf{\Gamma}, \leq_W)$, assuming $\mathrm{Det}(\mathbf{\Gamma})$, antichains have size at most 2.

Exercise 3. (Wadge reducibility)

Let $\mathbf{\Gamma}$ be a boldface pointclass closed under complements and intersections, and assume $\mathrm{Det}(\mathbf{\Gamma})$ (so for every $A, B \in \mathbf{\Gamma}$, the Wadge game $G^W(A, B)$ is determined). Let $\mathbf{\Delta}$ and $\mathbf{\Sigma}$ be other boldface pointclasses (not necessarily closed under complements), such that $\mathbf{\Delta} \subseteq \mathbf{\Gamma}$ and $\mathbf{\Sigma} \subseteq \mathbf{\Gamma}$.

Prove that either $\mathbf{\Delta} \subseteq \mathbf{\Sigma}$ or $\breve{\mathbf{\Sigma}} \subseteq \mathbf{\Delta}$.

(Here $\breve{\mathbf{\Sigma}} = \{A : \omega^\omega \setminus A \in \mathbf{\Sigma}\}$ is the dual pointclass).