A p-adic Birch and Swinnerton-Dyer conjecture for modular abelian varieties

Steffen Müller
Universität Hamburg
joint with Jennifer Balakrishnan and William Stein

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Notation

- $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \in S_2(\Gamma_1(N))$ newform,
- $K_f = \mathbb{Q}(\dots, a_n, \dots)$,
- $A_f = J_1(N)/\mathrm{Ann}_{\mathbb{T}}(f)J_1(N)$ abelian variety $/\mathbb{Q}$ associated to f,
- $g = [K_f : \mathbb{Q}]$ dimension of A_f,
- $G_f = \{\sigma : K_f \hookrightarrow \mathbb{C}\}$,
- $f^\sigma(z) = \sum_{n=1}^{\infty} \sigma(a_n) e^{2\pi i n z}$ for $\sigma \in G_f$,
- $L(A_f, s) = \prod_{\sigma \in G_f} L(f^\sigma, s)$ Hasse-Weil L-function of A_f.
- $L^*(A_f, 1)$ leading coefficient of series expansion of $L(A_f, s)$ in $s = 1$.

BSD conjecture

Conjecture (Birch, Swinnerton-Dyer, Tate)
We have $r := \mathrm{rk}(A_f(\mathbb{Q})) = \mathrm{ord}_{s=1} L(A_f, s)$ and

$$\frac{L^*(A_f, 1)}{\Omega^+_{A_f}} = \frac{\mathrm{Reg}(A_f/\mathbb{Q}) \cdot |Ш(A_f/\mathbb{Q})| \cdot \prod_v c_v}{|A_f(\mathbb{Q})_{\mathsf{tors}}| \cdot |A^\vee_f(\mathbb{Q})_{\mathsf{tors}}|}.$$

- $\Omega^+_{A_f}$: real period $\int_{A_f(\mathbb{R})} |\eta|$, η Néron differential,
- $\mathrm{Reg}(A_f/\mathbb{Q})$: Néron-Tate regulator,
- c_v: Tamagawa number at v, v finite prime,
- $Ш(A_f/\mathbb{Q})$: Shafarevich-Tate group.

Shimura periods

Let $p > 2$ be a prime such that A_f has good ordinary reduction at p.

We want to find a p-adic analogue of the BSD conjecture.

First need to construct a p-adic L-function.

Theorem. (Shimura) For all $\sigma \in G_f$ there exists $\Omega^+_{f^\sigma} \in \mathbb{C}^\times$ such that we have

(i) $\frac{L(f^\sigma,1)}{\Omega^+_{f^\sigma}} \in K_f$,

(ii) $\sigma\left(\frac{L(f,1)}{\Omega^+_f}\right) = \frac{L(f^\sigma,1)}{\Omega^+_{f^\sigma}}$,

(iii) an analogue of (i) for twists of f by Dirichlet characters.

Modular symbols

- Fix a Shimura period Ω_f^+.
- Fix a prime $\mathfrak{p}$ of K_f such that $\mathfrak{p} \mid p$.
- Let α be the unit root of $x^2 - a_p x + p \in (K_f)_{\mathfrak{p}}[x]$.
- For $r \in \mathbb{Q}$, the plus modular symbol is

$$[r]_f^+ := -\frac{\pi i}{\Omega_f^+}\left(\int_r^{i\infty} f(z)dz + \int_{-r}^{i\infty} f(z)dz\right) \in K_f.$$

- Define a measure on $\mathbb{Z}_p^\times$:

$$\mu_{f,\alpha}^+(a + p^n\mathbb{Z}_p) = \frac{1}{\alpha^n}\left[\frac{a}{p^n}\right]_f^+ - \frac{1}{\alpha^{n+1}}\left[\frac{a}{p^{n-1}}\right]_f^+$$

- Write $x \in \mathbb{Z}_p^\times$ as $\omega(x) \cdot \langle x \rangle$ where $\omega(x)^{p-1} = 1$ and $\langle x \rangle \in 1 + p\mathbb{Z}_p$.
- Define $L_p(f, s) := \int_{\mathbb{Z}_p^\times} \langle x \rangle^{s-1} \, d\mu^+_{f,\alpha}(x)$ for all $s \in \mathbb{Z}_p$.
- Fix a topological generator γ of $1 + p\mathbb{Z}_p$.
- Convert $L_p(f, s)$ into a p-adic power series $\mathcal{L}_p(f, T)$ in terms of $T = \gamma^{s-1} - 1$.
- Let $\epsilon_p(f) := (1 - \alpha^{-1})^2$ be the p-adic multiplier.

Then we have the following interpolation property (due to Mazur-Tate-Teitelbaum):

$$\mathcal{L}_p(f, 0) = L_p(f, 1) = \epsilon_p(f) \cdot [0]^+_f = \epsilon_p(f) \cdot \frac{L(f, 1)}{\Omega^+_f}.$$

Mazur-Tate-Teitelbaum conjecture

All of this depends on the choice of Ω_f^+!

If $A_f = E$ is an elliptic curve, a canonical choice is given by $\Omega_f^+ = \Omega_E^+$.

Conjecture. (Mazur-Tate-Teitelbaum) If $A_f = E$ is an elliptic curve, then we have $r := \mathrm{rk}(E/\mathbb{Q}) = \mathrm{ord}_{T=0}(\mathcal{L}_p(f,T))$ and

$$\frac{\mathcal{L}_p^*(f,0)}{\epsilon_p(f)} = \frac{\mathrm{Reg}_\gamma(E/\mathbb{Q}) \cdot |Ш(E/\mathbb{Q})| \cdot \prod_v c_v}{|E(\mathbb{Q})_{\mathrm{tors}}|^2}.$$

- $\mathcal{L}_p^*(f,0)$: leading coefficient of $\mathcal{L}_p(f,T)$,
- $\mathrm{Reg}_\gamma(E/\mathbb{Q}) = \mathrm{Reg}_p(E/\mathbb{Q})/\log(\gamma)^r$, where

  $\mathrm{Reg}_p(E/\mathbb{Q})$ is the p-adic regulator (due to Schneider, Néron, Mazur-Tate, Coleman–Gross, Nekovář).

Extending Mazur-Tate-Teitelbaum

An extension of the MTT conjecture to arbitrary dimension $g > 1$ should

- be equivalent to BSD in rank 0,
- reduce to MTT if $g = 1$,
- be consistent with the main conjecture of Iwasawa theory for abelian varieties.

Problem. Need to construct a p-adic L-function for A_f!

- Idea: Define $L_p(A_f, s) := \prod_{\sigma \in G_f} L_p(f^\sigma, s)$.
- But to pin down $L_p(f^\sigma, s)$, first need to fix a set $\{\Omega^+_{f^\sigma}\}_{\sigma \in G_f}$ of Shimura periods.

Theorem. If $\{\Omega^+_{f^\sigma}\}_{\sigma\in G_f}$ are Shimura periods, then there exists $c \in \mathbb{Q}^\times$ such that

$$\Omega^+_{A_f} = c \cdot \prod_{\sigma\in G_f} \Omega^+_{f^\sigma}.$$

- So we can fix Shimura periods $\{\Omega^+_{f^\sigma}\}_{\sigma\in G_f}$ such that

$$\Omega^+_{A_f} = \prod_{\sigma\in G_f} \Omega^+_{f^\sigma}. \tag{1}$$

- With this choice, define $L_p(A_f, s) := \prod_{\sigma\in G_f} L_p(f^\sigma, s)$.
- Then $L_p(A_f, s)$ does not depend on the choice of Shimura periods, as long as (1) holds.

Interpolation

- Convert $L_p(A_f, s)$ into a p-adic power series $\mathcal{L}_p(A_f, T)$ in terms of $T = \gamma^{s-1} - 1$.
- Let $\epsilon_p(A_f) := \prod_\sigma \epsilon_p(f^\sigma)$ be the p-adic multiplier.
- Then we have the following interpolation property

$$\mathcal{L}_p(A_f, 0) = \epsilon_p(A_f) \cdot \frac{L(A_f, 1)}{\Omega^+_{A_f}}.$$

More notation

- $\mathcal{L}_p^*(A_f, 0)$: leading coefficient of $\mathcal{L}_p(A_f, T)$,
- $\mathrm{Reg}_\gamma(A_f/\mathbb{Q}) = \mathrm{Reg}_p(A_f/\mathbb{Q})/\log(\gamma)^r$, where

  $r = \mathrm{rk}(A_f(\mathbb{Q}))$ and $\mathrm{Reg}_p(A_f/\mathbb{Q})$ is the p-adic regulator.
- If A_f is not principally polarized, then $\mathrm{Reg}_p(A_f/\mathbb{Q})$ is only defined up to ± 1.

We make the following p-adic BSD conjecture (with the obvious sign ambiguity if A_f is not principally polarized):

Conjecture. The Mordell-Weil rank r of $A_f/\mathbb{Q}$ equals $\mathrm{ord}_{T=0}(\mathcal{L}_p(A_f,T))$ and

$$\frac{\mathcal{L}_p^*(A_f,0)}{\epsilon_p(A_f)} = \frac{\mathrm{Reg}_\gamma(A_f/\mathbb{Q}) \cdot |Ш(A_f/\mathbb{Q})| \cdot \prod_v c_v}{|A_f(\mathbb{Q})_{\mathrm{tors}}| \cdot |A_f^\vee(\mathbb{Q})_{\mathrm{tors}}|}.$$

This conjecture

- is equivalent to BSD in rank 0,
- reduces to MTT if $g = 1$,
- is consistent with the main conjecture of Iwasawa theory for abelian varieties, via work of Perrin-Riou and Schneider.

Computing the p-adic L-function

To test our conjecture in examples, we need an algorithm to compute $\mathcal{L}_p(A_f, T)$.

- The modular symbols $[r]^+_{f^\sigma}$ can be computed efficiently in a purely algebraic way – up to a rational factor (Cremona, Stein),
- To compute $\mathcal{L}_p(A_f, T)$ to n digits of accuracy, can use
  - (i) approximation using Riemann sums (similar to Stein-Wuthrich) – exponential in n or
  - (ii) overconvergent modular symbols (due to Pollack-Stevens) – polynomial in n.
- Both methods are now implemented in Sage.

Normalization

To find the correct normalization of the modular symbols, can use the interpolation property.

- Find a Dirichlet character ψ associated to a quadratic number field $\mathbb{Q}(\sqrt{D})$ such that $D > 0$ and
  - $L(B,1) \neq 0$, where B is A_f twisted by ψ,
  - $\gcd(pN, D) = 1$.
- We have $\Omega_B^+ \cdot \eta_\psi = D^{g/2} \cdot \Omega_{A_f}^+$ for some $\eta_\psi \in \mathbb{Q}^\times$.
- Can express $[r]_B^+ := \prod_\sigma [r]_{f_\psi^\sigma}^+$ in terms of modular symbols $[r]_{f^\sigma}^+$.
- The correct normalization factor is

$$\frac{L(B,1)}{\Omega_B^+ \cdot [0]_B^+} = \frac{\eta_\psi \cdot L(B,1)}{D^{g/2} \cdot \Omega_{A_f}^+ \cdot [0]_B^+}.$$

Coleman-Gross height pairing

Suppose $A_f = \mathrm{Jac}(C)$, where $C/\mathbb{Q}$ is a hyperelliptic curve of genus g.

The Coleman-Gross height pairing is a symmetric bilinear pairing

$$h : \mathrm{Div}^0(C) \times \mathrm{Div}^0(C) \to \mathbb{Q}_p,$$

which can be written as a sum of local height pairings

$$h = \sum_v h_v$$

over all finite places v of $\mathbb{Q}$ and satisfies $h(D, \mathrm{div}(g)) = 0$ for $g \in k(C)^\times$.

Techniques to compute h_v depend on v:

- $v \neq p$: intersection theory (M., Holmes)
- $v = p$: logarithms, normalized differentials, Coleman integration (Balakrishnan-Besser)

Local heights away from p

- $D, E \in \mathrm{Div}^0(C)$ with disjoint support,
- suppose $v \neq p$,
- $X\,/\,\mathrm{Spec}(\mathbb{Z}_v)$: regular model of C,
- $(\ .\ )_v$: intersection pairing on X,
- $\mathcal{D}$, $\mathcal{E} \in \mathrm{Div}(X)$: extensions of D, E to X such that $(\mathcal{D}\,.\,F)_v = (\mathcal{E}\,.\,F)_v = 0$ for all vertical divisors $F \in \mathrm{Div}(X)$.
- Then we have
$$h_v(D, E) = -(\mathcal{D}\,.\,\mathcal{E})_v \cdot \log_p(v).$$

Computing local heights away from p

- Regular models can be computed using `Magma`;
- divisors on C and extensions to X can be represented using Mumford representation;
- intersection multiplicities of divisors on X can be computed algorithmically using linear algebra and Gröbner bases (M.) or resultants (Holmes).
- All of this is implemented in `Magma`.

Local heights at p

Let ω_D be a normalized differential associated to D. The local height pairing at p is given by

$$h_p(D,E) = \int_E \omega_D.$$

- Suppose $C/\mathbb{Q}_p$ is given by an odd degree model $y^2 = g(x)$.
- Let P, $Q \in C(\mathbb{Q}_p)$.
- If $P \equiv Q \pmod{p}$, then it is easy to compute $\int_P^Q \omega_D$.
- The work of Balakrishnan-Besser gives a method to extend this to the rigid analytic space $C_{\mathbb{C}_p}^{\mathrm{an}}$, using analytic continuation along Frobenius.
- Need to compute matrix of Frobenius, e.g. using Monsky-Washnitzer cohomology.
- This has been implemented by Balakrishnan in Sage.

Computing the p-adic regulator

- Suppose $P_1, \dots, P_r \in A_f(\mathbb{Q})$ are generators of $A_f(\mathbb{Q})$ mod torsion.
- Suppose $P_i = [D_i]$, $D_i \in \mathrm{Div}(C)^0$ pairwise relatively prime and with pointwise $\mathbb{Q}_p$-rational support.
- Then $\mathrm{Reg}_p(A_f/\mathbb{Q}) = \det((m_{ij})_{i,j})$, where $m_{ij} = h(D_i, D_j)$.

Problem. Given a subgroup H of $A_f(\mathbb{Q})$ mod torsion of finite index, need to saturate it.

- Currently only possible for $g = 2$ ($g = 3$ work in progress due to Stoll), so in general only get $\mathrm{Reg}_p(A_f/\mathbb{Q})$ up to a $\mathbb{Q}$-rational square.
- For $g = 2$, can use generators of H and compute the index using Néron-Tate regulators to get $\mathrm{Reg}_p(A_f/\mathbb{Q})$ exactly.

Empirical evidence for $g = r = 2$

- From "Empirical evidence for the Birch and Swinnerton-Dyer conjectures for modular Jacobians of genus 2 curves" (Flynn et al. '01), we considered 16 genus 2 curves of respective level N.
- For each curve, the associated abelian variety has Mordell-Weil rank 2.

N	Equation
67	$y^2 + (x^3 + x + 1)y = x^5 - x$
73	$y^2 + (x^3 + x^2 + 1)y = -x^5 - 2x^3 + x$
85	$y^2 + (x^3 + x^2 + x)y = x^4 + x^3 + 3x^2 - 2x + 1$
93	$y^2 + (x^3 + x^2 + 1)y = -2x^5 + x^4 + x^3$
103	$y^2 + (x^3 + x^2 + 1)y = x^5 + x^4$
107	$y^2 + (x^3 + x^2 + 1)y = x^4 - x^2 - x - 1$
115	$y^2 + (x^3 + x^{+}1)y = 2x^3 + x^2 + x$
125	$y^2 + (x^3 + x + 1)y = x^5 + 2x^4 + 2x^3 + x^2 - x - 1$
133	$y^2 + (x^3 + x^2 + 1)y = -x^5 + x^4 - 2x^3 + 2x^2 - 2x$
147	$y^2 + (x^3 + x^2 + x)y = x^5 + 2x^4 + x^3 + x^2 + 1$
161	$y^2 + (x^3 + x + 1)y = x^3 + 4x^2 + 4x + 1$
165	$y^2 + (x^3 + x^2 + x)y = x^5 + 2x^4 + 3x^3 + x^2 - 3x$
167	$y^2 + (x^3 + x + 1)y = -x^5 - x^3 - x^2 - 1$
177	$y^2 + (x^3 + x^2 + 1)y = x^5 + x^4 + x^3$
188	$y^2 = x^5 - x^4 + x^3 + x^2 - 2x + 1$
191	$y^2 + (x^3 + x + 1)y = -x^3 + x^2 + x$

$N = 188$

To numerically verify p-adic BSD, need to compute p-adic regulators and p-adic special values. For example, for $N = 188$, we have:

p-adic regulator $\mathrm{Reg}_p(A_f/\mathbb{Q})$	p-adic L-value	p-adic multiplier $\epsilon_p(A_f)$
$5623044 + O(7^8)$	$1259 + O(7^4)$	$507488 + O(7^8)$
$4478725 + O(11^7)$	$150222285 + O(11^8)$	$143254320 + O(11^8)$
$775568547 + O(13^8)$	$237088204 + O(13^8)$	$523887415 + O(13^8)$
$1129909080 + O(17^8)$	$6922098082 + O(17^8)$	$4494443586 + O(17^8)$
$14409374565 + O(19^8)$	$15793371104 + O(19^8)$	$4742010391 + O(19^8)$
$31414366115 + O(23^8)$	$210465118 + O(23^8)$	$45043095109 + O(23^8)$
$2114154456754 + O(37^8)$	$1652087821140 + O(37^8)$	$1881820314237 + O(37^8)$
$6279643012659 + O(41^8)$	$2066767021277 + O(41^8)$	$4367414685819 + O(41^8)$
$9585122287133 + O(43^8)$	$3309737400961 + O(43^8)$	$85925017348 + O(43^8)$
$3328142761956 + O(53^8)$	$5143002859 + O(53^6)$	$6112104707558 + O(53^8)$
$17411023818285 + O(59^8)$	$7961878705 + O(59^6)$	$98405729721193 + O(59^8)$
$102563258757138 + O(61^8)$	$216695090848 + O(61^7)$	$137187998566490 + O(61^8)$
$26014679325501 + O(67^8)$	$7767410995 + O(67^6)$	$38320151289262 + O(67^8)$
$490864897182147 + O(71^8)$	$16754252742 + O(71^6)$	$530974572239623 + O(71^8)$
$689452389265311 + O(73^8)$	$193236387 + O(73^5)$	$162807895476311 + O(73^8)$
$878760549863821 + O(79^8)$	$1745712500 + O(79^5)$	$1063642669147985 + O(79^8)$
$2070648686579466 + O(83^8)$	$2888081539 + O(83^5)$	$1103760059074178 + O(83^8)$
$3431343284115672 + O(89^8)$	$1591745960 + O(89^5)$	$1012791564080640 + O(89^8)$
$4259144286293285 + O(97^8)$	$21828881 + O(97^4)$	$6376229493766338 + O(97^8)$

$N = 188$ – normalization

The additional BSD invariants for $N = 188$ are

$$|\text{Ш}(A_f)[2]| = 1,\ |A_f(\mathbb{Q})_{\text{tors}}|^2 = 1,\ c_2 = 9,\ c_{47} = 1.$$

We find that for the quadratic character ψ associated to $\mathbb{Q}(\sqrt{233})$, the twist B of A_f by ψ has rank 0 over $\mathbb{Q}$.

- Algebraic computation yields $[0]_B^+ = 144$,
- $\eta_\psi = 1$, computed by comparing bases for the integral 1-forms on the curve and its twist by ψ.
- $\frac{\eta_\psi \cdot L(B,1)}{233 \cdot \Omega_{A_f}^+} = 36$.
- So the normalization factor for the modular symbol is $1/4$.

Summary of evidence

Theorem. Assume that for the Jacobians of all 16 curves listed above the Shafarevich-Tate group over $\mathbb{Q}$ is 2-torsion. Then our conjecture is satisfied up to least 4 digits of precision at all good ordinary $p < 100$ satisfying the assumptions of our algorithms.

- The assertion $Ш(A_f/\mathbb{Q}) = Ш(A_f/\mathbb{Q})[2]$ follows from classical BSD (Flynn et al.).
- We also have a similar result for the Jacobian of a twist of $X_0(31)$ of rank 4 over $\mathbb{Q}$.
- Since the twist is odd, we had to use the minus modular symbol associated to $J_0(31)$.