1. Suppose that the entries of $A \in \mathbb{R}^{n\times n}$ are non-negative and the sum of any row or column is 1. Show that A is a convex combination of permutation matrices (i.e. quadratic matrices where every row and column is a unit vector).

2. Let T be (the edge set of) a tree with $r \in V(T)$. Consider the hypergraph $\mathcal{H} := \{P \subseteq T \,:\, P \text{ is a path starting at } r\}$. Show that the incidence matrix of $\mathcal{H}$ is TU.

3. Let $G = (V, E)$ be a bipartite graph with minimal degree k. Show that there is a partition $F_1, \dots, F_k$ of E such that (V, F_i) has no isolated vertices.

4. Let $G = (V, E)$ be a bipartite graph. Give a necessary and sufficient condition for the existence of $F, H \subseteq E$ with $d_H(v) = d_F(v) + 1$ for every $v \in V$.

5. Let $D = (V, A)$ be a digraph. An $x \in \mathbb{R}^A$ is called a **circulation** if $\varrho_x(v) = \delta_x(v)$ for $v \in V$. Let $f, g \in \mathbb{R}^A$ with $f \le g$ be given. Circulation x is **feasible** if $f \le x \le g$. Show that there is a feasible circulation if and only if $\varrho_f(Z) \le \delta_g(Z)$ holds for every $Z \subseteq V$.

Hints for the hard direction: call an edge e **tight** (with respect to D, f, g) if $f(e) = g(e)$. Similarly $Z \subseteq V$ is tight if $\varrho_f(Z) = \delta_g(Z)$.

1. Start with a hypothetical counterexample D, f, g, where f, g are chosen for D in such a way that the sum of the number of tight sets and tight edges is as large as possible.
2. Show that there is an edge e which is not tight.
3. Let $d_{g-f}(X, Y)$ be the sum of the values of $(g - f)$ on the edges between $X \setminus Y$ and $Y \setminus X$ (both direction). Let $\beta(Z) := \delta_g(Z) - \delta_f(Z)$ and prove the following equation by checking that the contribution of any edge is the same for both sides:
$$\beta(X) + \beta(Y) = \beta(X \cup Y) + \beta(X \cap Y) + d_{g-f}(X, Y) \quad (\forall X, Y \subseteq V).$$
4. Check that e leaves a tight set X and enters a tight set Y and get a contradiction applying the previous equation and the fact that e is not tight.

6. Prove that if $f, g \in \mathbb{Z}^A$ in the previous exercise, then the feasible circulation can be chosen integral.