1. Let $C \subseteq \mathbb{R}^n$ be convex and suppose that $x \in C$ is a convex combination of some elements of C where none of them is x itself. Show that x is a convex combination of two such elements as well.

2. Prove that for a nonempty polyhedron $P = P(Q, b)$ the following sets are the same:

1. $\{z \mid \forall x \in P\ \forall \lambda \in \mathbb{R}^+\ x + \lambda z \in P\}$,
2. $\{z \mid \exists x \in P\ \forall \lambda \in \mathbb{R}^+\ x + \lambda z \in P\}$,
3. $\{z \mid Qz \leq \underline{0}\}$.

3. Let P_1, P_2 be polyhedra, is it true that $P_1 + P_2 := \{x + y : x \in P_1,\ y \in P_2\}$ is a polyhedron as well?

4. The convex hull of finitely many (at least one) vectors is called a **polytope**. Show that every polytope is a polyhedron. (* Is it true that exactly the bounded polyhedra are the polytopes?)

5. Let $A \in \mathbb{R}^{m \times n}$ and suppose that $b \in G(A)$. Prove that there is an A' that consists of at most n rows of A for which $b \in G(A')$.

6. Derive the following general form of the Farkas lemma:

$$\nexists x = (x_0, x_1):\ x_1 \geq \underline{0},\ Ax_0 + Bx_1 = b_0,\ Cx_0 + Dx_1 \leq b_1 \text{ iff}$$
$$\exists y = (y_0, y_1):\ y_1 \geq \underline{0},\ y_0 A + y_1 C = \underline{0},\ y_0 B + y_1 D \geq \underline{0},\ y_0 b_0 + y_1 b_1 < 0.$$

7. According to Helly's theorem if $C_1, \ldots, C_m$ are convex sets in $\mathbb{R}^n$ and every $n+1$ intersects, then $\bigcap_{i=1}^m C_i \neq \varnothing$.

- Show that the theorem follows easily from the special case where C_i are polytopes.
- *Prove the theorem for polyhedra using Farkas lemma and exercise 5.

8.* Let $D = (V, A)$ be a finite digraph and $c : A \to \mathbb{R}$. The cost function c is called **conservative** if there is no directed cycle C with $\sum_{uv \in C} c(uv) < 0$. We call a function $\pi : V \to \mathbb{R}$ a potential. It is feasible if $\forall uv \in A:\ \pi(v) - \pi(u) \leq c(uv)$.

- Show that there is a feasible potential if and only if c is conservative.
- Does this remain true in infinite digraphs?
- Show that if c is conservative and t is reachable from s, then the cost of a cheapest $s \to t$ path is

$$\max\{\pi(t) - \pi(s) : \pi \text{ is a feasible potencial}\}.$$