1. Let $D = (V, A)$ be a strongly connected digraph and let $c \in \mathbb{Z}_+^A$ and $w \in \mathbb{Z}_+^V$. A multiset of vertices (i.e. a $x \in \mathbb{Z}_+^V$) is called **c-independent** if for every directed cycle K we have $x(V(K)) \leq c(K)$ (here $x(V(C)) = \sum_{v \in V(K)} x(v)$ and $c(K) = \sum_{e \in C} c(e)$). A multiset of directed cycles **covers** w if for every $v \in V$ there are at least $w(v)$ directed cycles in the multiset that goes through v. Prove the following minimax theorem:

$$\max\{xw : x \text{ is } c\text{-independent}\} = \min\{\sum_{K \in \mathcal{Y}} c(K) : \mathcal{Y} \text{ is a multiset of directed cycles covering } w\}$$

Hint:

1. Write it as a linear program and its dual.
2. Show by splitting every vertex to a directed edge that the dual is essentially a minimal cost feasible circulation problem.
3. Show that this circulation problem has an integral optimal solution and hence the original dual as well.
4. Conclude that the primal system is TDI and hence there is an integral optimal solution (since the matrix and the bounding vector are integral).

2. Let $D = (V, A)$ be a strongly connected digraph and let $\alpha(D) := \max\{|U| : U \subseteq V, U \text{ spans no edges in } D\}$. Prove that one can cover the vertices of D with at most $\alpha(D)$ directed cycles.

Hint: Pick an $F \subseteq A$ such that F covers the directed cycles in D and for each $e \in A$ there is some directed cycle K such that $e \in K$ and $|F \cap K| = 1$. (We have seen in some earlier exercise that such an F exists.) Apply the minimax theorem at the previous exercise with $w \equiv 1$ on V and with $c := \chi_F$ (i.e. let c be a characteristic function of F).