1. A **tree representation** of a set family $\mathcal{F} \subseteq 2^V$ consists of a directed tree $T = (U, F)$ and a function $\varphi : V \to U$ for which the inverse images of the sets U_{uv} (here $uv \in F$ and U_{uv} is the vertex set of the component of $T - uv$ that contains v) with respect to φ are exactly the elements of $\mathcal{F}$ i.e., $\mathcal{F} = \{\varphi^{-1}(U_{uv}) : uv \in F\}$.

A family $\mathcal{F} \subseteq 2^V$ is called **laminar** if for any $X, Y \in \mathcal{F}$ at least one of the following relations holds: $X \subseteq Y,\ X \supseteq Y,\ X \cap Y = \varnothing$. Prove that any laminar system has a tree representation in which T is an arborescence.

2. Let $Qx \leq b,\ Px = d$ be a TDI system. Replace the first inequality $Q_{1.}x \leq b_1$ by $Q_{1.}x = b_1$. Prove that if the new system is still solvable, then it is TDI as well. (Hint: Check first the special case when the modified dual has an optimal solution in which $y_{Q_1} \geq 0$. What is the connection between the set of optimal dual solutions with respect to a $c \in \mathbb{Z}^n$ and with respect to $c + nQ_{1.}$ for some $n \in \mathbb{N}$ in the modified system?)

3. Let $D = (V + r, A)$ be a digraph in which $\mathsf{in}_A(r) = \varnothing$ and there is a directed path from r to any $v \in V$. We are looking for a cheapest spanning r-arborescence with respect to a given $c \in \mathbb{R}_+^A$. Show that the following algorithm gives one.

1. If there is a nonempty $Z \subseteq V$ such that $c(e) > 0$ for $e \in \mathsf{in}_A(Z)$, then take a $\subseteq$-minimal such a Z and decrease c on the elements of $\mathsf{in}_A(Z)$ by $\min\{c(e) : e \in \mathsf{in}_A(Z)\}$. Continue this with the modified c. Iterate as long as possible.
2. If every nonempty $Z \subseteq V$ has an ingoing 0-edge (edge with $c(e) = 0$), then build the arborescence in the following way. Start with the trivial arborescence consists of the root r. If we have already an r-arborescence $(U + r, F)$ with $U \subsetneq V$, then extend it with an element f of $\{e \in \mathsf{in}_A(V \setminus U) : c(e) = 0\} := O$ that became a 0-edge earlier or at the same time as the other edges in O during the first phase of the algorithm.

Hint: Let $Q \in \{0,1\}^{2^V \times A}$ be a matrix where the entry corresponding to a (Z, e) is 1 if $e \in \mathsf{in}_A(Z)$ and 0 otherwise. The algorithm finds actually an optimal solution of $\min cx\ Qx \geq \underline{1},\ x \geq \underline{0}$. To see this let $y \equiv 0$ on 2^V at the beginning and when we pick a $Z \subseteq V$ in the first phase increase $y(Z)$ by $\min\{c(e) : e \in \mathsf{in}_A(Z)\}$. Check that y remains a dual solution and the final version y^* of y and the characteristic function x^* of the edge set F^* of the final r-arborescence satisfy the following optimality criteria:

- for $e \in F$, we have $c(e) = \sum_{e \in \mathsf{in}_A(Z)} y(Z)$, (actually it holds for every 0-edge)
- if $y(Z) > 0$, then $\mathsf{in}_{F^*}(Z) = 1$ (the $\subseteq$-minimal choice of Z is important here).

Conclude from this that x^* and y^* are primal and dual optimal solutions respectively. (Furthermore, if c is integral, then $y^* : 2^V \to \mathbb{Z}_+$).

4. Let $D = (V + r, A)$ as in the previous exercise. Prove that

$$
\begin{aligned}
& x \in \mathbb{R}^A \\
& x \geq \underline{0} \\
& \varrho_x(Z) \geq 1 \quad (Z \subseteq V,\ |Z| > 1) \\
& \varrho_x(v) = 1 \quad (v \in V)
\end{aligned}
$$

is a TDI system and the polytope of the solutions is exactly the convex hull of the characteristic vectors of the spanning r-arborescences of D. (Hint: Use the two previous exercises)