1. Prove that if D and D' are strongly connected orientations of $G = (V, E)$ with $\varrho_D(v) = \varrho_{D'}(v)$ for $v \in V$, then one can get D' from D by reversing directed cycles repeatedly.

2. Let $G = (V, E)$ be a 2-edge-connected graph and $f \in \mathbb{N}^V$. For $X \subseteq V$, let us denote the number of connected components of $G - X$ by $\sigma(X)$. Prove that there is a strongly connected orientation D of G with $\varrho_D(v) \geq f(v)$ for $v \in V$ if and only if $f(X) + \sigma(X) \leq e(X)$ for every nonempty $X \subseteq V$ (where $f(X) := \sum_{v \in X} f(v)$ and $e(X)$ is the number of edges with at least one endpoint in X).

Hint:

1. Show first the necessity of the condition: $f(X) + \sigma(X) \leq e(X)$ for every nonempty $X \subseteq V$.
2. Take an arbitrary strongly connected orientation D' of G and suppose that $\varrho_{D'}(u) < f(u)$.
3. Consider the set $\mathcal{Y}$ of the $\subseteq$-maximal elements of $\{Y \subseteq V - u : \varrho_{D'}(Y) = 1\}$. Prove that the elements of $\mathcal{Y}$ are disjoint and there is no edge between them.
4. If there is a $w \in V \setminus \bigcup \mathcal{Y}$ with $\varrho_{D'}(w) > f(w)$, then reverse a $u \to w$ path of D'.
5. If there is no such a w then show that $X := V \setminus \bigcup \mathcal{Y}$ violates the condition.

3. Let $V := \{v_1, \ldots, v_n\}$ and $A := \{v_i v_j : 1 \leq i < j \leq n\}$. What is the largest amount of the $v_1 \to v_n$ flows with respect to the capacity $g(v_i v_j) := j - i$?

4. Let $D = (V, A)$ be a digraph with $s \neq t \in V$ and let $g : A \to \mathbb{R}^+$ be given. Show that the set

$$\mathcal{O} := \{U \subseteq V : t \in U, s \notin U, \varrho_g(U) = \lambda_g(s, t)\}$$

is closed under union and intersection (where $\lambda_g(s, t)$ is the value of a minimal $s \to t$ cut).

5. Derive Hall's theorem (about matchings in bipartite graphs) from the Maxflow-Mincut theorem.