Modern techniques in combinatorial set theory

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Chapter 1

Introduction

What single ideas lead to the breakthrough results of infinite combinatorics in the last 30 years? And which methods found the widest range of applications? The main purpose of this course is to overview novel set-theoretic and combinatorial techniques. We will especially focus on constructing uncountable structures with prescribed combinatorial behaviour.

We will first focus on *inductive constructions and diagonalisation arguments.* In general, such construction follow a similar scheme: one enumerates all the requirements on the final structure and then proceeds by induction, building an increasingly large object that satisfies more and more requirements. Starting from simple examples, we look at more complex constructions that require special care when choosing the initial enumeration. Often, this is based on choosing a sequence of *elementary submodels* containing all relevant parameters. We will introduce this technique in detail and cover various applications from the set theory of Euclidean spaces, graph theory and general topology.

The second main topic of the course is centred around the following fact: there is a sequence of injective maps $e_\alpha : \alpha \to \omega$ for $\alpha < \omega_1$ which are *coherent* i.e., e_α and e_β disagree at at most finitely many values. We will arrive at such maps through S. Todorcevic's theory of *minimal walks on ordinals.* Our goal is to cover the basic properties of walks and see how this ties into the classification of uncountable linear orders and Ramsey theory on small uncountable cardinals (in particular, on $\aleph_1$ and $\aleph_2$).

Finally, we will survey further *general construction schemes.* We introduce tools that support inductively building large objects by one small piece at a time. We will cover trees of elementary submodels; $\diamondsuit$-type axioms and other guessing principles; Kurepa families; and Todorcevic's construction scheme.

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1.1 Notation and preliminaries.

The set of finite numbers/ordinals $\{0, 1, 2 \ldots\}$ is denoted by ω. We work with von Neumann ordinals i.e., α is the set of all smaller ordinals. In particular, if we write $f : \mathbb{R} \to 2$ then f is a map from the set of reals to $\{0, 1\}$.

We use $\mathfrak{c}$ or $2^{\aleph_0}$ to denote the cardinality of $\mathbb{R}$. We will regularly use the fact that a set of cardinality $\mathfrak{c}$ has $\mathfrak{c}$ many countable subsets i.e., $|[\mathfrak{c}]^\omega|= \mathfrak{c}$. We often identify the cardinal 2^κ with the set of all functions from κ to 2. The notation $[X]^\kappa$ stands for the collection of κ sized subsets of X. We let X^κ denote the set of all functions from κ to X (i.e., κ-tuples).

1.1.1 Basic set theory

A set $C \subset \kappa$ is a *club* if $\sup C = \kappa$ (unbounded) and for any $\alpha \in \kappa$, $\sup(C \cap \alpha) = \alpha$ implies $\alpha \in C$ (closed). A set $S \subset \kappa$ is *stationary* if S has non-empty intersection with any club subset of κ. The club sets form a filter that is closed under taking intersections of size $< \kappa$. You might think of them as having measure 1 in some sense. Stationary sets behave like sets of positive measure. They do not form a filter and in fact, κ can be partitioned into κ many pairwise disjoint stationary sets.

Lemma 1.1.1 (Fodor's pressing down lemma)**.** *Suppose that κ is a regular cardinal and $S \subseteq \kappa$ is stationary. If $f : S \to \kappa$ is regressive i.e., for any $\alpha \in S \setminus \{0\}$, $f(\alpha) < \alpha$, then there is a stationary $T \subset S$ so that $f \restriction T$ is constant.*

1.1.2 Basic graph and Ramsey theory

A graph G is a pair (V, E) of vertices V and edges E so that $E \subset [V]^2$. An *independent set* in G is a set of vertices with no edges, a *clique or complete subgraph* is a set of vertices with all edges between distinct vertices.

A graph H is a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. Whenever we say that H *is contained in* G we mean that G has a subgraph isomorphic to H. A subgraph H is *induced* if $E(H) = [V(H)]^2 \cap E(G)$.

The *chromatic number of a graph* G, usually denote by $\chi(G)$, is the minimal number of independent vertex sets that can cover G. In other words, the minimal (finite or infinite) cardinal r so that there is a colouring $c : V \to r$ such that there are no monochromatic edges.

Let us summarise the most basic infinite Ramsey results with the so-called arrow notation:

1. (F. P. Ramsey 1930, [35]) For any finite n and k,

$$\omega \to (\omega)^n_k$$

   i.e., for any colouring $c : [\omega]^n \to k$ there is an infinite $A \subset \omega$ so that $c \restriction [A]^n$ is constant.

In general, the left-hand side of the arrow denotes the base set of the colouring. On the right-hand side, the upper index is the dimension (are we colouring singletons, pairs, triples?), the lower index the number of colours and inside the bracket you see the size or order-type of the monochromatic sets we can always select. If the arrow is crossed over that means there is a colouring witnessing without monochromatic sets of that particular size. Let us continue with some further examples.

2. (Erdős-Dushnik-Miller, 1941) For any infinite κ,

$$\kappa \to (\kappa, \omega)^2_2$$

i.e., for any colouring $c : [\kappa]^2 \to 2$ either there is an $A \subset \kappa$ of size κ so that $c \restriction [A]^2$ is constant 0 or there is an infinite $A \subset \kappa$ so that $c \restriction [A]^2$ is constant 1.

3. (Erdős-Rado, 1956 [13]) For any infinite κ,

$$(2^\kappa)^+ \to (\kappa^+)^2_\kappa$$

i.e., for any colouring $c : [(2^\kappa)^+]^2 \to \kappa$ there is an $A \subset \kappa$ of size κ^+ so that $c \restriction [A]^2$ is constant.[1]

Two famous negative partition results that are good to keep in mind are the following.

4. (Sierpinski, 1933 [39]) For any $\lambda \leq \mathfrak{c}$,

$$\lambda \not\to (\omega_1)^2_2$$

i.e., there is a colouring $c : [\lambda]^2 \to 2$ so that both colours appear on any uncountable subset of λ.

5. (Todorcevic, 1987 [45])

$$\omega_1 \not\to [\omega_1]^2_{\omega_1}$$

i.e., there is a colouring $c : [\omega_1]^2 \to \omega_1$ so that all ω_1 many colours appear on any uncountable subset of ω_1.

Suppose that $\mathcal{F}$ is a family of sets. The chromatic number of $\mathcal{F}$ is the minimal r so that there is a colouring $c : \bigcup \mathcal{F} \to r$ such that $c \restriction a$ is not constant for any $a \in \mathcal{F}$. In the arrow notation, we could write

$$\bigcup \mathcal{F} \not\to (\mathcal{F})^1_r.$$

If $\mathcal{F}$ is the edge-set of a graph then we get back the usual notion of graph chromatic number.

1.1.3 General resources

Classical textbooks in set theory [28, 20, 18]; A. Rinot has a great blog about combinatorial set theory;[2] the standard textbook in general topology [9].

[1] This theorem has an analogue for colouring n-tuples but one needs to iterate the exponential function $n-1$ times.

[2] See here http://blog.assafrinot.com/

Chapter 2

Inductive constructions and closure arguments

2.1 Simple diagonalisations

Let us warm up with a simple diagonalisation argument for constructing an interesting subset of $\mathbb{R}$.

Theorem 2.1.1 (W. Sierpinski, 1932 [38]). *There is a dense and rigid $X \subset \mathbb{R}$.*

Proof. Our goal is to construct X in such a way that there is no non-trivial continuous bijection $g : X \to X$. While there are $2^{\mathfrak{c}}$ many maps $X \to X$ in general, there are only continuum many continuous functions $X \to X$. Indeed, any continuous map is uniquely determined by its values on a countable, dense subset X_0 of X.[1]

We will start with the set of rationals $X_0 = \mathbb{Q}$. Given some continuous $f : \mathbb{Q} \to \mathbb{R}$, there is a unique largest $X_f \subset \mathbb{R}$ and continuous $\bar{f} : X_f \to \mathbb{R}$ so that $f \subset \bar{f}$. Indeed, X_f consists of all $x \in \mathbb{R}$ so that

$$\bigcap_{n \in \omega} f[\mathbb{Q} \cap (x - \frac{1}{n}, x + \frac{1}{n})]$$

is a singleton y, in which case, we put $\bar{f}(x) = y$. Now, list all $f : \mathbb{Q} \to \mathbb{R}$ which are

1. not the identity, and
2. $X_f \cap I$ has size continuum for any interval I

Waclaw Sierpinski 1882-1969

as $\{f_\alpha : \alpha < \mathfrak{c}\}$.

We shall construct two sequences of points $\{x_\alpha : \alpha < \mathfrak{c}\}$ and $\{y_\alpha : \alpha < \mathfrak{c}\}$, each pairwise distinct, so that $\bar{f}_\alpha(x_\alpha) = y_\alpha$. The idea is that we will promise to put x_α into X but not y_α and so eliminating f_α as a (restriction of a) potential homeomorphism of X. To make sure X has size continuum in each interval in the end, we keep a list $\{J_\alpha : \alpha < \mathfrak{c}\}$ of all non-empty intervals, each enumerated $\mathfrak{c}$-times.

[1] So the number of continuous maps is bounded by $|X|^{|X_0|} = \mathfrak{c}^{\aleph_0} = \mathfrak{c}$.

How does the induction go? Given $\{x_\alpha : \alpha < \beta\}$ and $\{y_\alpha : \alpha < \beta\}$, we look at f_β. This function is not the identity so we can find $q \in \mathbb{Q}$ with $q \neq f_\beta(q)$. Let I denote an open interval around q so that $I \cap \bar{f}_\beta[I] = \emptyset$. Pick

$$x_\beta \in I \cap X_{f_\beta} \setminus \{x_\alpha, y_\alpha, \bar{f}_\beta^{-1}(x_\alpha), \bar{f}_\beta^{-1}(y_\alpha) : \alpha < \beta\}$$

which is possible since X_{f_β} was assumed to have size continuum in each interval. Now, $y_\beta = \bar{f}_\beta(x_\beta) \notin \{x_\alpha, y_\alpha : \alpha < \beta\}$ as desired. We also select an extra point $z_\beta \in J_\beta$ that is distinct from all the x_α and y_α points (for $\alpha \leq \beta$) we chose so far.

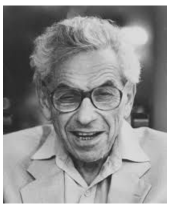
Now, we can let $X = \mathbb{Q} \cup \{x_\alpha, z_\alpha : \alpha < \mathfrak{c}\}$ and this set will satisfy the requirements. That is, if $g : X \to X$ is continuous but not the identity then we let $f = g \upharpoonright \mathbb{Q}$ and note that X_f must satisfy the two requirements above (since $g \subset \bar{f}$) and so there is some $\alpha < \mathfrak{c}$ so that $g \upharpoonright \mathbb{Q} = f_\alpha$. In turn, $g \subset \bar{f}_\alpha$ which implies $g(x_\alpha) = \bar{f}_\alpha(x_\alpha) = y_\alpha \notin X$, a contradiction. □

There are two features of the above argument especially relevant for us: first, we managed to take care of all relevant $X \to X$ maps by realising that each such map is completely characterised by its behaviour on a countable set. Second, no matter which enumeration $\{f_\alpha : \alpha < \mathfrak{c}\}$ we take, the construction goes through (producing different sets possibly). This is not always the case, however. Often, we need a special enumeration to avoid getting stuck in the construction of our desired structure. The next result will make these remarks clear.

Theorem 2.1.2 (P. Erdős and A. Hajnal, 1969 [11])**.** *There is a colouring $c : \mathbb{R}^2 \to \omega$ so that no two points of the same colour are in rational distance.*

We will call such colourings good in the following proof.

Proof. Let's do something less ambitious first: take some countable $X \subset \mathbb{R}^2$ and find a good colouring $c : X \to \omega$. Of course this is trivial as we can make sure that no two points are of the same colour. Now take some $y \in \mathbb{R}^2 \setminus X$ and try to extend c by defining $c(y)$. The problem is that maybe, for any possible colour $n < \omega$ that we can choose for y, there is already an $x \in X$ with $c(x) = n$ and $|x - y| \in \mathbb{Q}$. This hints that if we define c inductively then the domains should be large enough to contain these y's at bad distances from old points. Let's see the actual proof now.



Paul Erdős
1913-1996

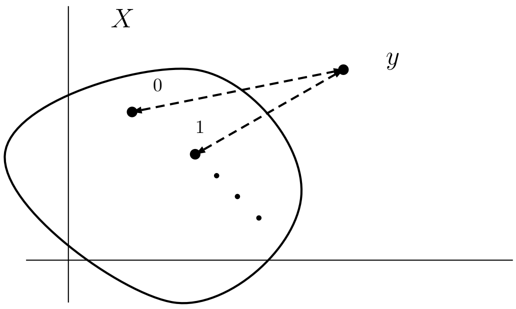


Figure 2.1: Extending a partial colouring.

We will say that $X \subset \mathbb{R}^2$ is *closed enough* if whenever $x \neq x'$ are both in X and $|x - y|, |x' - y|$ are both rational then $y \in X$ too. That is, for any $y \notin X$, there is at most one $x \in X$ of rational distance to y.

Claim 2.1.3. *For any infinite $X \subset \mathbb{R}^2$, there is a closed enough $\hat{X} \supseteq X$ of the same size.*

Proof. Define a map F by

$$F(x, x') = \bigcup_{q,q' \in \mathbb{Q}} (\{y : |x - y| = q\} \cap \{y : |x - y| = q'\}).$$

Note that $F(x, x')$ is a countable union of intersections of two circles and hence is countable. Now, starting from $X_0 = X$, let $X_{n+1} = X_n \cup \bigcup\{F(x, x') : x \neq x' \in X_n\}$. Then $\hat{X} = \bigcup_{n\in\omega} X_n$ is closed enough and has the same cardinality as X. $\square$

We prove that any closed enough $X \subset \mathbb{R}^2$ has a good colouring $c : X \to \omega$ by induction on $\kappa = |X|$. For $\kappa = \mathfrak{c}$ and $X = \mathbb{R}^2$, this gives the theorem. Again, the $\kappa = \aleph_0$ case is trivial.

Now, given X of size κ we use the claim to write $X = \bigcup_{\alpha<\mathrm{cf}(\kappa)} X_\alpha$ so that X_α is closed enough of size $< \kappa$ and the sequence is continuous i.e., $X_\beta = \bigcup_{\alpha<\beta} X_\alpha$ for any limit $\beta < \mathrm{cf}(\kappa)$.

Now, define c along this decomposition inductively. A good map $c_0 : X_0 \to \omega$ must exist because X_0 has size less than κ. Given c_α for $\alpha < \beta$, if β is limit we simply take $c_\beta = \cup_{\alpha<\beta} c_\alpha$.

In the successor case, when $\beta = \alpha + 1$, we take a good colouring d on $X_{\alpha+1} \setminus X_\alpha$ which maps into $\{2n : n < \omega\}$. Note that $c_\alpha \cup d$ might not be a good choice (why?). But if $y \in X_{\alpha+1} \setminus X_\alpha$ then there is at most one $x \in X_\alpha$ of rational distance from y. So look at the two values $\{d(y), d(y)+1\}$ and pick $c_{\alpha+1}(y) \in \{d(y), d(y)+1\} \setminus \{c_\alpha(x)\}$. This new $c_{\alpha+1}$ will have no conflict with points in X_α and remains a good colouring on $X_{\alpha+1} \setminus X_\alpha$ as well. This finishes the inductive construction of $c : X \to \omega$ and the proof of the theorem.

$\boxtimes$

This theorem can be nicely rephrased in the language of graphs. We can define a graph $G_\mathbb{Q}$ on $\mathbb{R}^2$ by taking xy to be an edge if and only if $|x - y| \in \mathbb{Q}$. Now the theorem states the existence of a countable colouring so that each colour class is independent i.e., that the graph has *countable chromatic number*. Note that for any set of distances $D \subset \mathbb{R}$, one can define a graph G_D similarly. One of the most well-studied examples is the unit distance graph $G_{\{1\}}$ and the chromatic number of this graph is famously unknown to this day.

Open Problem 2.1.4 (Hadwiger-Nelson)**.** *What is the minimal number of sets avoiding distance 1 that can cover the plane?*

The chromatic number of $G_{\{1\}}$ is known to be either 5, 6 or 7 (for the latest result see [7]). For the classical lower bound of 4 and upper bound of 7, see Figure 2.2 below.

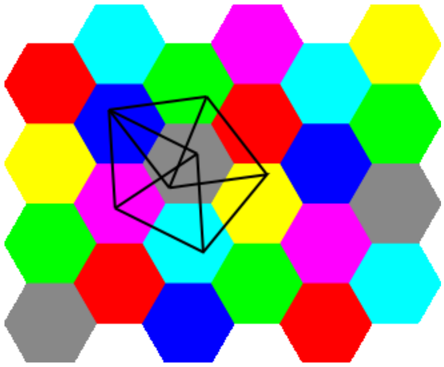


Figure 2.2: The Moser spindle (of chromatic number 4) and a 7-colouring of the plane.

For a survey of this problem and similar fun questions see the book [40].

Summary

We saw two simple inductive proofs: in both cases, we constructed something in $\mathfrak{c}$ steps while taking care of $\mathfrak{c}$ many objectives. While in the first case, the enumeration of the objectives was arbitrary, in the second proof we needed a carefully chosen list. This was provided by a closure argument. A useful trick to keep in mind: the behaviour of large objects are often determined by their traces on small substructures (e.g., continuous functions on $\mathbb{R}$ are uniquely characterised by their values on $\mathbb{Q}$).

Exercises and problems

Exercise 2.1.5. *Starting with a single coin, you play a game with a simple automaton: at each step you insert a single coin to which the machine returns two coins.*

1. *Show that an unassuming player might loose all his/her money in ω steps.*
2. *Show that, with any strategy, the player will go bankrupt in countably many steps.*

Exercise 2.1.6. *Suppose that $A \subseteq \mathbb{C}$ is arbitrary. Show that there is an algebraically closed subfield $F \subseteq \mathbb{C}$ of size at most $|A|+\aleph_0$ which contains A.*

Exercise 2.1.7. *Suppose that $f_\alpha : \omega_1 \to \omega_1$ for $\alpha < \omega_1$. Show that there is a club $C \subset \omega_1$ such that for any $\beta \in C$ and $\alpha, \xi < \beta$, $f_\alpha(\xi) < \beta$ as well.*

Exercise 2.1.8. *Suppose that each line ℓ in $\mathbb{R}^2$ is assigned a natural number $m_\ell \geq 2$. Construct a set $A \subset \mathbb{R}^2$ which meets each line ℓ in exactly m_ℓ points.*

Problem 2.1.9. *Show that $\mathbb{R}^2$ has a well order $\prec$ so that for any y, the set $\{x \prec y : |y-x| \in \mathbb{Q}\}$ is finite. Why does it follow that the rational distance graph on $\mathbb{R}^2$ has countable chromatic number?*

Problem 2.1.10. *Prove that the family of all non-empty perfect subsets of* $\mathbb{R}$ *has chromatic number 2. In fact, show that there is a function* $g : \mathbb{R} \to \mathbb{R}$ *so that* $g[C] = \mathbb{R}$ *for any copy* C *of the Cantor set in* $\mathbb{R}$.

Problem 2.1.11. *Prove that any graph* G *of size at most continuum is* spatial *i.e., there is an injective* $f : V(G) \to \mathbb{R}^3$ *so that for any pair of edges* $ab, cd \in E(G)$*, the open line segments* $(f(a), f(b))$ *and* $(f(c), f(d))$ *are disjoint.*

Problem 2.1.12. *Can we cover* $\mathbb{R}^2$ *by disjoint circles? How about* $\mathbb{R}^3$*? Can you do it by unit circles only?*

Problem 2.1.13. *Show that* $\mathbb{R}^+$ *can be decomposed into two, disjoint sets both closed under addition.*

Challenge 2.1.14. *Show that the rational distance graph on* $\mathbb{R}^3$ *has countable chromatic number too.*

⋆

Open Problem 2.1.15 (Erdős)**.** *Is there a Borel set* $A \subset \mathbb{R}^2$ *which meets each line* ℓ *in exactly 2 points?*

Open Problem 2.1.16 (Ulam)**.** *Does there exist a dense set* $S \subseteq \mathbb{R}^2$ *so that all pairwise distances between points in* S *are rational?*

Open Problem 2.1.17. *Is there a Borel partition of* $\mathbb{R}^3$ *by unit circles?*

Further reading

Set theory in Euclidean spaces [24, 14]; diagonalisations of length continuum and general topology [46, Chapter 4]; results and problems on colouring [40].

2.2 Working with elementary submodels

The idea behind elementary submodels is very simple: given a large structure $\mathfrak{A}$, you would like to consider substructures $\mathfrak{B}$ which are smaller than $\mathfrak{A}$ but reflect basic properties of its original structure. Suppose $\mathfrak{A}$ is some Euclidean space $A = \mathbb{R}^n$ (which is of size continuum) along with lines, planes, hyperplanes, etc. Now, you look for a small set of points $B \subset A$ together with a small set of lines, planes, etc. that satisfy the same relations as A and its lines, planes, etc. That is, if two lines of $\mathfrak{B}$ meet in $\mathbb{R}^n$ then they must meet in B and in turn, their unique intersection must be in B as well. Similarly, for any three points in B, there is a (hyper)plane in $\mathfrak{B}$ that contains them (since there was one in $\mathfrak{A}$). If this hyperplane is not unique, then there could be ones which are in $\mathfrak{A}$ but not in $\mathfrak{B}$. If there is only a small number of new objects definable from a given set of elements already in B then we can throw in all those without increasing the size our structure. Repeat this process and, if you keep track of all objects and operations appropriately, you will end up with the desired substructure.

Whenever you occur any similar situation, you can carry out a closure argument, starting from some fixed B_0, adding more and more points and objects inductively. The general framework of elementary submodels will provide a tool which saves you from repeating the very same closure argument over and over and give you the nicest substructures you can imagine, all in a single step.

2.2.1 The basics

Models and formulas. Recall that the language $\mathcal{L}$ of set theory only involves a single binary relation $\in$. We are allowed to build formulae using $\in$, the logical symbols $\{\exists, \forall, (,), \neg, \wedge\}$ and (countably many) variables. There is a simple inductive procedure that yields all possible formulas, starting from the elementary formula $x \in y$.

Let V denote the set theoretic universe, the class of all sets, and we assume that V satisfies the usual ZFC axioms. We will mostly be interested in $\mathcal{L}$-structures of the form $(M, \in)$ where M itself is a set from V; we often just write M to mean $(M, \in)$. What does it mean that a formula holds in a model M? Given $\varphi(x_0, \dots, x_n)$ and $a_0, \dots, a_n \in M$, we can define $M \models \varphi(a_0, \dots, a_n)$ by induction on φ.

For the elementary formula $x \in y$, we write $M \models x \in y$ if actually $x \in y$ in V holds. Suppose that $\phi(x) = (\forall y)\varphi(x)$. Then

$$(M, \in) \models (\forall y)\varphi(y, x)$$

if and only if for any $y \in M$, $M \models \varphi(y, x)$. Similarly, if $\phi(x) = (\exists y)\varphi(y, x)$ then

$$(M, \in) \models (\exists y)\varphi(y, x)$$

if and only if there is some $y \in M$ so that $M \models \varphi(y, x)$.

Of course, an arbitrary set M as an $\in$-structure has no reason to share too many properties of the set theoretic universe. In other words, M and V will satisfy very different formulas.

The $H(\theta)$ models and reflection. Now, let θ be a cardinal. We use $H(\theta)$ to denote the collection of sets of hereditary cardinality $< \theta$; in other words, sets x which have transitive closure of size $< \theta$.[2] So, for example, $H(\aleph_0)$ is the collection of hereditarily finite sets: each $a \in H(\aleph_0)$ is finite, moreover, any element $b \in a$ is finite, and any element $c \in b$ is finite, and so on.

We collected some properties of these models here.

Fact 2.2.1. *For any infinite θ,*

1. *$H(\theta)$ is transitive i.e., $b \in a \in H(\theta)$ implies $b \in H(\theta)$;*
2. *$H(\theta)$ has size $2^{<\theta}$ (in particular, its a set);*
3. *$H(\theta) \cap ON = \theta$ (where ON denotes the class of all ordinals);*
4. *if θ is uncountable and regular then all of the ZFC axioms except the power set axiom is satisfied by the model $(H(\theta), \in)$.*

Let us also mention that this $H(\theta)$ hierarchy agrees with the V_θ hierarchy (based on the power-set operation) on a closed and unbounded set of ordinals.

Now, if $x \in H(\theta)$ and $2^{|x|} < \theta$ then the power set $\mathcal{P}(x) = \{y : y \subset x\}$ is also in $H(\theta)$. So, choosing θ large enough ensures that any argument, with limited iterations of the power set operation, can be carried out in $H(\theta)$ instead of the whole set-theoretic universe.

Furthermore, compared to V, $H(\theta)$ is not any other model of set theory (minus the powerset). These models, for all practical purposes, completely reflect the behaviour of the underlying universe.

Theorem 2.2.2. *Given a finite set of formulae Σ and cardinal ρ, there is a $\theta > \rho$ so that for any $\varphi \in \Sigma$ and $a_0, \ldots, a_n \in H(\theta)$:*

$$H(\theta) \models \varphi(a_0, \ldots, a_n) \text{ if and only if } \varphi(a_0, \ldots, a_n) \text{ is true in the universe } V.$$

This means that whatever theorem or formula $\varphi(a_1, \ldots, a_n)$ you are trying to prove, it suffices to check its validity in models of the form $H(\theta)$ where θ is large enough to contain $a_1, \ldots, a_n$ and φ *is absolute between* $H(\theta)$ *and* V (i.e., satisfies the equation in the last theorem). We also say that $H(\theta)$ is a Σ-elementary submodel of V. When you read "let θ be large enough" without any further explanation, the authors usually mean the above choice.

Countable elementary submodels. So we found some relatively small models that resemble the whole universe but what we need is models of arbitrary size (often countable) that can still 'talk' about large structures (say the real line). This is certainly not true for the $H(\theta)$ models.

A submodel of $(H(\theta), \in)$ is simply a structure of the form $(M, \in)$ so that $M \subset H(\theta)$. Suppose that $x \in M$ and $\phi(x)$ is some formula with parameter(s) x from M. What does $(M, \in) \models \phi(x)$ mean again?

If $\phi(x) = (\exists y)\varphi(y, x)$ then

$$(M, \in) \models (\exists y)\varphi(y, x)$$

[2] The transitive closure $tc(x)$ of a set x is defined as $tc(x) = \cup\{tc(x, n) : n \in \omega\}$ where $tc(x, 0) = x$ and $tc(x, n+1) = \cup tc(x, n)$.

if and only if there is some $y \in M$ so that $M \models \varphi(y, x)$. Think about ϕ saying that two given lines x_0, x_1 (in M) have *an* intersection y. Now, $(M, \in) \models (\exists y)\varphi(y, x)$ means that M contains such an intersection. Of course this is a unique point so *the* intersection is in M.

Note that it becomes harder to satisfy $(M, \in) \models (\exists y)\varphi(y, x)$ than to satisfy $(H(\theta), \in) \models (\exists y)\varphi(y, x)$ since for M, we have less y to choose from.

Similarly, let $\phi(x) = (\forall y)\varphi(x)$. Then

$$(M, \in) \models (\forall y)\varphi(y, x)$$

if and only if for any $y \in M$, $M \models \varphi(y, x)$. So if M is a proper subset of $H(\theta)$ then at first sight, $(M, \in) \models (\forall y)\varphi(y, x)$ is easier to satisfy than $(H(\theta), \in) \models (\forall y)\varphi(y, x)$ since for $H(\theta)$ we have more y to consider.

M being an elementary submodel of $H(\theta)$ is a sort of equilibrium point for the above satisfactions.

Definition 2.2.3. *We say that $(M, \in)$ is an elementary submodel $(H(\theta), \in)$ if $M \subset H(\theta)$ and for any first-order formula $\phi(x)$ and parameters a from M,*

$$(M, \in) \models \phi(a) \text{ if and only if } (H(\theta), \in) \models \phi(a).$$

We write $M \prec H(\theta)$ in this case (understanding that elementarity is over the $\in$ relation).

In other words, the structures $(M, \in)$ and $(H(\theta), \in)$ completely agree about formulas that concern objects in M. For example, $M = H(\theta)$ is a valid but uninteresting choice of an elementary submodel.

By Gödel's Second Incompleteness Theorem, we cannot prove the existence of elementary submodels of V in using ZFC alone. In contrast, we can always take elementary submodels of sets (but not classes) by the downward Löwenheim-Skolem theorem [28, Theorem I.15.10].

Theorem 2.2.4 (Löwenheim-Skolem)**.** *For any set $A \subseteq H(\theta)$, there is an elementary submodel M with $A \subseteq M$ and $|M| = |A| + \aleph_0$.*

So any infinite set $A \subset H(\theta)$ can be included in an elementary submodel $M \prec H(\theta)$ of the same size. We will not cover the proof; it is a standard closure argument (inductively adding more and more witnesses for existential formulas) included in many classical textbooks.

To reiterate, the exact value of θ in our arguments will never play a role, we just assume that θ is large enough so that $H(\theta)$ includes all relevant parameters that our important for our purposes and is Σ-elementary in V for an appropriately large set of formulas (based on the result we are trying to prove). If we prove something about $\mathbb{R}$, then $\mathfrak{c}^+$ or $2^{\mathfrak{c}}$ is a good choice for θ usually. If you prove a theorem about subsets of an $\aleph_{27}$-dimensional vector space over $\mathbb{F}_{11}$ then pick $\theta = (2^{\aleph_{27}})^+$.

2.2.2 Some useful facts

Now that we saw that elementary submodels exist, lets see what makes them so useful. First, if $x, y \in M \prec H(\theta)$ then $(x, y), x \cup y, x \cap y \in M$. Moreover, for any function $f \in M$ with $x \in \mathrm{dom}(f)$, $f(x) \in M$ as well.

Let's take now a non-empty, countable $M \prec H(\aleph_2)$. As a first step, let's see how does $M \cap ON$ look like. It is easy to see that $\emptyset \in M$ and so $\omega \subset M$ follows easily. Even more, $\omega \in M$ since ω is uniquely definable in $H(\aleph_2)$. Similarly, $\omega_1 \in M$ as well since in $H(\aleph_2)$, we can uniquely define ω_1 as the smallest cardinal above ω. Note that $\aleph_2 \notin M$ since $\aleph_2$ is not even in $H(\aleph_2)$.

However, since M is countable, ω_1 cannot be a subset of M. Countable elementary submodels are far from being transitive i.e., $\underline{A \in M \text{ does not imply } A \subseteq M \text{ in general}}$. However, we will see that $M \cap \omega_1$ is an initial segment of ω_1.

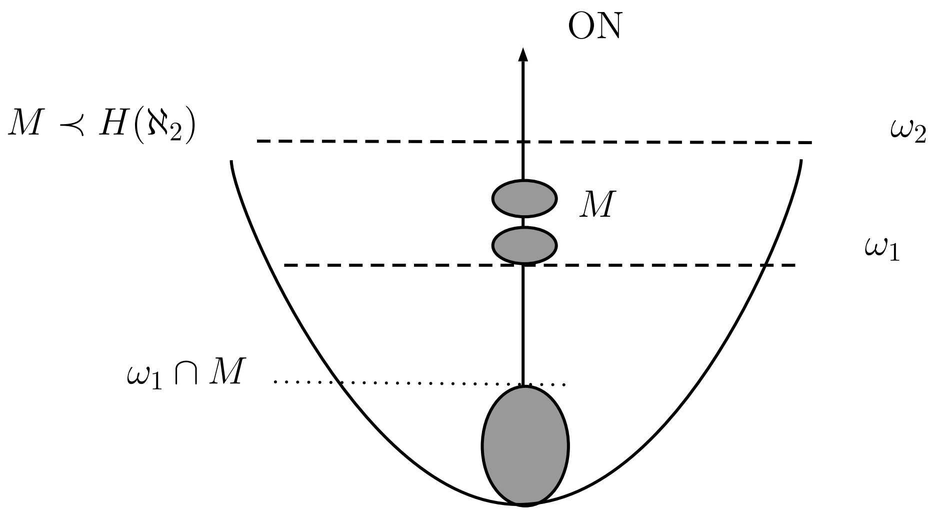


Figure 2.3: $H(\aleph_2)$ and a countable elementary submodel

However, if $x \in M$ is finite then $x \subset M$ does hold. We will apply this and the following fact regularly.

Main Fact 2.2.5. *Suppose that $M \prec H(\theta)$ is a countable elementary submodel (for some $\theta \geq \aleph_2$) and $X \in M$.*

1. *If X is countable then $X \subset M$;*
2. *if $X \setminus M \neq \emptyset$ then X must be uncountable;*
3. *$M \cap \omega_1$ is an initial segment of ω_1;*
4. *if $X \subset \omega_1$ is uncountable then $X \cap M$ is a cofinal subset of $\omega_1 \cap M$.*

Similarly, if μ is a cardinal which is an element and subset of M and $X \in M$ has size μ then $X \subset M$ as well.

Proof. Note that the first two statements are equivalent so we prove (1) only. If X is countable then the formula

$$\phi(X) \equiv (\exists f : \omega \to X) f[\omega] = X$$

must hold in $H(\theta)$. So, $\phi(X)$ holds in M as well and hence we can pick $f : \omega \to X$, $f \in M$ so that $f[\omega] = X$. Now, for any $n \in \omega$, both $n, f \in M$ so $f(n) \in M$ as well. In turn, $X = f[\omega] = \{f(n) : n \in \omega\} \subset M$ as desired.

Now, (3) follows easily: if $\beta \in M \cap \omega_1$ then β is of course countable so $\beta \subset M$ i.e., for any $\alpha < \beta$, $\alpha \in M$ as well.

Finally, suppose that there is some $\alpha \in M \cap \omega_1$ so that for any $x \in X \cap M$, $x < \alpha$. In turn,

$$M \models (\forall x \in X) x < \alpha$$

so $H(\theta)$ must satisfy the same formula. That is, we must have $x < \alpha$ for any $x \in X$ (since $H(\theta)$ and V agree about this). However, this contradicts that X was uncountable. □

The following observation is quite useful as well and points to the fact that for elementary submodels M, the ordinal $M \cap \omega_1$ plays a critical role.

Fact 2.2.6. *Suppose that $M \prec H(\theta)$ is countable and $\alpha = M \cap \omega_1$. If $X \in M$ is a subset of ω_1 and $\alpha \in X$ then X is stationary.*

Conversely, for any stationary $S \subset \omega_1$ and $X \in H(\theta)$, there is a countable $M \prec H(\theta)$ which contains X and $M \cap \omega_1 \in S$.

Proof. First, take a club $C \subset \omega_1$ so that $C \in M$. By the previous fact, $C \cap \omega_1$ is cofinal in M and in turn, $\alpha \in C$ as well (because C is closed). So $X \cap C \neq \emptyset$. We just showed that the statement 'for any club $C \subset \omega_1$, $X \cap C \neq \emptyset$' is true in M. So this must be true in $H(\theta)$ as well, which in turn, implies that X is stationary.

Second, we claim that there is a continuous, increasing sequences of models $M_\alpha \prec H(\theta)$ all containing X so that $\omega_1 \subset \cup\{M_\alpha : \alpha < \omega_1\}$ (see the exercises). Now, $\{M_\alpha \cap \omega_1 : \alpha < \omega_1\}$ must be closed and unbounded so there is an $\alpha < \omega_1$ so that $\omega_1 \cap M_\alpha \in S$.[3] □

We will return to sequences and chains of elementary submodels in later sections.

Corollary 2.2.7 (Fodor's pressing down lemma)**.** *Suppose T is stationary and $f : T \to \omega_1$ is regressive i.e., $f(\alpha) < \alpha$ for any $\alpha \in T \setminus \{0\}$. Then f is constant on a stationary subset of T.*

Proof. Pick some countable $M \prec H(\aleph_2)$ so that $f \in M$ and $\alpha = M \cap \omega_1 \in T$. Note that $\varepsilon = f(\alpha) < \alpha$ so $\varepsilon \in M$. In turn, if we let $S = f^{-1}(\varepsilon) \subset T$ then $\alpha \in S \in M$ and so S must be a stationary set as well. By definition, f is constant ε on S. □

Extended languages. Finally, we mention that it is often useful to extend the model $(H(\theta), \in)$ with additional relations and predicates. For example, we can take a well-order $<_w$ of $H(\theta)$ and consider elementary submodels M of the structure $(H(\theta), \in, <_w)$. The benefit of doing this is the following: we often build structures, such as graphs or tree orders, inductively along chains of elementary submodels. At each stage, we can usually find many equally good choices that satisfy our requirements (say to extend a tree with a new level). If we use the $<_w$ well-order to pick a *minimal* good choice then we construct a unique object that only depends on the sequence of models. In turn, a larger model M_γ that contains a restricted sequence $(M_\alpha)_{\alpha<\beta}$ can re-create the construction up to level β of the object solely in M_γ. This comes handy in various situations.

[3] Since $\{\alpha < \omega_1 : \alpha = M_\alpha \cap \omega_1\}$ is also a club, we can also assume that $\alpha = M_\alpha \cap \omega_1 \in S$.

Exercises and problems

Exercise 2.2.8. *Prove that $H(\aleph_1)$ satisfies the statement 'all sets are countable.'*

Exercise 2.2.9. *Show that the transitive closure of any $a \in H(\theta)$ is again an element of $H(\theta)$. Show that $[H(\theta)]^{<\theta} \subset H(\theta)$.*

Exercise 2.2.10. *Suppose that $(M_\alpha)_{\alpha<\mu}$ is an increasing sequence of elementary submodels of some $H(\theta)$. Prove that $M = \bigcup\{M_\alpha : \alpha < \mu\}$ is also an elementary submodel of $H(\theta)$.*

Exercise 2.2.11. *Show that if μ is a cardinal which is an element and subset of M and $X \in M$ has size μ then $X \subset M$ as well.*

Exercise 2.2.12. *Show that any uncountable $A \subset \mathbb{R}$ contains uncountably many complete accumulation points i.e., $x \in A$ so that $|U \cap A| = |A|$ for any open neighbourhood U of x.*

Exercise 2.2.13. *Suppose that $M \prec H(\theta)$ is some elementary submodel and $\kappa \in M$. When is $\kappa \cap M$ an initial segment of κ?*

Problem 2.2.14. *Prove that, for large enough θ, for any $X \subset H(\theta)$ of size $\mathfrak{c}$ there is a countably closed elementary submodel $M \prec H(\theta)$ of size $\mathfrak{c}$ which contains X.*

Problem 2.2.15. *For any uncountable $A \subset \mathbb{R}^n$, there is an uncountable $B \subset A$ such that all distances between points in B are pairwise different.*

Problem 2.2.16. *Find a Borel function $f : \mathbb{R} \to \mathbb{R}$ with the property that $f[I] = \mathbb{R}$ for any non-empty open $I \subset \mathbb{R}$.*

Problem 2.2.17. *Show that $\mathbb{R}$ is the union of countably many sets $(A_i)_{i<\omega}$ so that none of the A_i contains a 3-element arithmetic progression.*

Problem 2.2.18. *Suppose that G is a graph and $k < \omega$. Assume that any finite subgraph of G has chromatic number at most k. Prove that $\chi(G) \leq k$ as well.*

2.2.3 The first applications

We start by a standard example of demonstrating the use of elementary submodels. A family of sets F is called a Δ-system (or sunflower) if there is a single r so that for any $a \neq b \in F$, $r = a \cap b$. This set r is called the root or core of the Δ-system.

Theorem 2.2.19 (Δ-system lemma)**.** *Every uncountable family of finite sets contains an uncountable Δ-system.*

The Δ-system lemma is one of the most cited results in set theory, ubiquitous in forcing arguments, topological and Ramsey results.

Proof. We can assume, without loss of generality, that we work with subsets of ω_1 i.e. take an uncountable $F \subset [\omega_1]^{<\omega}$. Pick a countable $M \prec H(\aleph_2)$ so that $F \in M$. Fix any $b \in F \setminus M$ and let $r = b \cap M$. Note that $r \in M$ and so the set

$$E = \{a \in F : r \subset a\}$$

is also an element of M. Moreover, $b \in E \setminus M$ and so E must be uncountable. We shall find our uncountable Δ-system in E with root r.

Indeed, we take a maximal subfamily E_0 of E which satisfies that $\{a \setminus r : a \in E_0\}$ is pairwise disjoint i.e., E_0 is a Δ-system with root r. Moreover, we pick such an E_0 in M.

We claim that E_0 must be uncountable. Otherwise, if E_0 is countable then $E_0 \subset M$ and each element $a \in E_0$ is a subset of M. In turn, $E_0 \cup \{b\} \subset E$ is a proper extension of E_0 which still forms a Δ-system with root r. While the set $E_0 \cup \{b\}$ is not an element of M, it still witnesses (in $H(\theta)$) that E_0 is not maximal. So, by elementarity, E_0 cannot be maximal in M either, which contradicts our initial maximal choice of E_0.

Finally, note that E_0 is the desired uncountable Δ-system.

$\boxtimes$

Most often, we use two types of elementary submodels: countable models and *countably closed models* of size $\mathfrak{c}$. The latter means that if $x \subset M$ is countable then $x \in M$ as well. In some sense, this is the reverse of the transitivity property we proved in the Main Fact. The existence of countably closed models of size $\mathfrak{c}$ is the $\mu = \omega$ case of the following fact.

Fact 2.2.20. *For any μ and $X \subset H(\theta)$, there is $M \prec H(\theta)$ so that $X \subset M$, $|M| = |X|^\mu$ and M is μ-closed i.e., $[M]^\mu \subset M$.*

We omit the proof which is again a variant of the Löwenheim-Skolem closure argument.

We shall make use of countably closed elementary submodels of size continuum now. Recall that these models satisfy the property that if $x \subset M$ is countable then $x \in M$ as well. The following theorem was an answer to a five-decade-old problem of Alexandroff and Urysohn whether there is a compact, first countable space with cardinality greater than the continuum.[4] Recall that first countable means that any point has a countable neighbourhood base. A space is compact if any open cover has a finite subcover; we assume that compact spaces are Hausdorff i.e., any two points have disjoint open neighbourhoods. This implies that convergent sequences have unique limits, as expected in any reasonable space.

[4] The proof we present is due to R. Pol.

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Theorem 2.2.21 (A. Arhangel'skii, 1969 [9]). *Every compact, first countable topological space (X, τ) has size at most continuum.*

We only use the fact that if x is in the closure of a set A then there is a convergent subsequence of A with limit x.

Proof. Take a countably closed model $M \prec H(\theta)$ of size continuum with $(X, \tau) \in M$. Since the topology is in M, we can use formulas that talk about subsets of X (which are elements of M) being closed, open, convergent, etc. In particular, if $A \in M$ is a subset of X then the closure $\bar{A}$ with respect to τ is also an element of M.

Claim 2.2.22. *$X \cap M$ is closed and so compact.*

Proof. If $x \in \overline{X \cap M}$ then, since X is first countable, there is a sequence $A = \{x_n : n \in \omega\} \subset M \cap X$ converging to x. I.e., x is the unique accumulation point of A. However, $A \in M$ as M is countably closed and so the unique accumulation point x of A is also in M. Indeed, $\bar{A} = A \cup \{x\} \in M$ is countable so $\bar{A} \subset M$ by the Main Fact. Hence, $\overline{X \cap M} \subseteq X \cap M$ as desired. $\square$

To finish the proof of the theorem, it suffices to prove the following.

Claim 2.2.23. *$X \cap M = X$.*

Proof. Suppose that $y \in X \setminus M$. Note that for any $x \in M \cap X$, there is a countable neighbourhood base $\mathcal{B}_x \in M$. By countability, $\mathcal{B}_x$ is also a subset of M. Now, for any $x \in M$, select $U_x \in \mathcal{B}_x$ so that $x \in U_x \subset X \setminus \{y\}$. The open family $\{U_x : x \in X \cap M\}$ covers the compact space $X \cap M$ and so there is a finite subfamily $\{U_x : x \in F\}$ that covers $X \cap M$. The latter finite family is an element of M and so $M \models (\forall x' \in X)x' \in \{U_x : x \in F\}$. In turn, $\{U_x : x \in F\}$ must really cover X which contradicts $y \notin \{U_x : x \in F\}$. $\square$

$\boxtimes$

In fact, Arhangel'skii proved that for any Hausdorff space X, the following inequality holds:

$$|X| \leq 2^{\chi(X) \cdot L(X)}.$$

Here, $\chi(X)$ is the minimal cardinal κ so that any point in X has a neighbourhood base of size at most κ, and $L(X)$ is the minimal λ so that any open cover has a subcover of size at most λ. So, in the previous theorem, we essentially proved the case of $\chi(X) = L(X) = \aleph_0$.[5]

2.2.4 Chains of elementary submodels

Elementary submodels are often used to cut a large structure X into smaller pieces. The literature refers to these as filtrations sometimes.

Fact 2.2.24. *Suppose that $X \in H(\theta)$ is of size κ and $\operatorname{cf} \kappa = \mu$. Then there is a sequence $(M_\alpha)_{\alpha<\mu}$ so that*

1. *$X \in M_\alpha \prec H(\theta)$ and $|M_\alpha| < \kappa$,*

[5] Compactness is stronger than assuming $L(X) = \aleph_0$ which is referred to as being Lindelöf.

2. *(continuity) for any limit* $\beta < \mu$, $M_\beta = \cup\{M_\alpha : \alpha < \beta\}$,

3. $X \subset \bigcup\{M_\alpha : \alpha < \mu\}$.

Moreover, we can assume that $(M_\alpha)_{\alpha<\beta} \in M_{\beta+1}$ *for all* $\beta < \mu$.

We use the models M_α to write X as the increasing union of $X \cap M_\alpha$. We often apply some inductive assumption to $X \cap M_\alpha$ or $X \cap M_{\alpha+1} \setminus M_\alpha$. Note that the latter sets partition X because we assumed that the sequence $(M_\alpha)_{\alpha<\mu}$ is continuous.

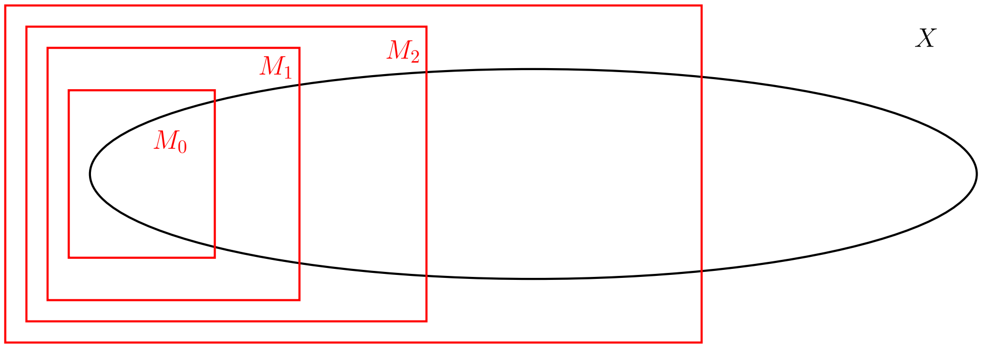

Figure 2.4: A filtration of a structure X

Moreover, note that if κ is regular then $\{\kappa \cap M_\alpha : \alpha < \kappa\}$ is a club subset of κ.

Let's see an application of chains of models. First, recall how we found uncountable Δ-systems in families of finite sets. Note that you can always make a Δ-system pairwise disjoint by removing the root (this is often used in applications). Is the same true for an arbitrary set system with pairwise small intersections? The next theorem addresses this question, but first some definitions. We say that

- a family of sets F is *d-almost disjoint* if $|x \cap y| < d$ for any $x, y \in F$;
- F is *essentially disjoint* if for any $x \in F$, there is a finite $f(x)$ so that $\{x \setminus f(x) : x \in F\}$ is pairwise disjoint.

Theorem 2.2.25 (P. Komjáth, 1984 [22])**.** *Suppose that d is finite. Then every d-almost disjoint family of countable sets is essentially disjoint.*

The theorem is optimal in the following sense: first, note that any uncountable, almost disjoint $F \subset [\omega]^\omega$ is not essentially disjoint. One can also find almost disjoint families of countable sets which are *locally countable* i.e., for any countable A, $\{x \cap A : x \in F\}$ is countable, but F is still not essentially disjoint. On the other hand, there are 2-almost disjoint families of sets of size continuum which are not essentially disjoint (see the exercises).

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Proof. Let us fix a family $\mathcal{A}$ of countable sets so that $x \cap y$ has size $< d$ for any $x \neq y \in \mathcal{A}$. We need to find a map $x \mapsto f(x)$ so that $f(x)$ is finite and $\{x \setminus f(x) : x \in \mathcal{A}\}$ is pairwise disjoint. In fact, we will show that there is a well order $<$ on $\mathcal{A}$ so that

$$y \cap \bigcup\{x \in \mathcal{A} : x < y\}$$

is finite for any $y \in \mathcal{A}$. This clearly suffices since we can let $f(y) = y \cap \bigcup\{x \in \mathcal{A} : x < y\}$.

Now, we prove the existence of the well order by induction on the size of $\mathcal{A}$. If $\mathcal{A}$ is countable then any type ω well order works just fine.

In general, assume $\mathcal{A}$ has size κ. Take a continuous sequence of elementary submodels $(M_\alpha)_{\alpha<\mathrm{cf}(\kappa)}$, each of size $< \kappa$ and containing $\mathcal{A}$ as an element. Note that each $\mathcal{A}_\alpha = \mathcal{A} \cap M_{\alpha+1} \setminus M_\alpha$ has a good well order $<_\alpha$ by the inductive assumption. Also, $\mathcal{A}$ is partitioned by the $\mathcal{A}_\alpha$ sets so it would be great if we could just glue these $<_\alpha$ orders together. This is possible by the next claim.

Claim 2.2.26. *If $y \in \mathcal{A}_\beta$ then both $y \cap M_\beta$ and $y \cap \bigcup_{\alpha<\beta} \mathcal{A}_\alpha$ are finite.*

Proof. Actually, $y \cap M_\beta$ has less than d elements. Otherwise, take $y_0 \subset y \cap M_\beta$ of size exactly d. Note that $\{y' \in \mathcal{A} : y_0 \subset y'\} \in M_\beta$ and $\{y' \in \mathcal{A} : y_0 \subset y'\} = \{y\}$ must hold by d-almost disjointness. In turn, $y \in M_\beta$, contradicting $y \in \mathcal{A}_\beta = \mathcal{A} \cap M_{\beta+1} \setminus M_\beta$.

Finally, note that $\bigcup_{\alpha<\beta} \mathcal{A}_\alpha$ is a subset of M_β. □

Now, define $<$ on $\mathcal{A}$ as follows: take $x \in \mathcal{A}_\alpha, y \in \mathcal{A}_\beta$ and let $x < y$ if either $\alpha < \beta$ or $\alpha = \beta$ and $x <_\alpha y$. This order $<$ on $\mathcal{A}$ is as desired.

⊠

Exercises and problems

Exercise 2.2.27. *Let d be a finite number. Prove that any infinite family of d-element sets contains an infinite Δ-system.*

Suppose that κ is an infinite cardinal. The basic open sets in the product topology on 2^κ are of the form $[\varepsilon] := \{f \in 2^\kappa : \varepsilon \subset f\}$ where ε is a finite partial function from κ to 2.

Exercise 2.2.28. *Let κ be an infinite cardinal. Show that there is no uncountable family of pairwise disjoint non-empty open subsets of 2^κ.*[6]

A topological space (X, τ) is called separable if it has a countable dense subset. (X, τ) is said to be Lindelöf if any open cover of X has a countable subcover.

Exercise 2.2.29. *Suppose that (X, τ) is a topological space and $(X, \tau) \in M \prec H(\theta)$. Prove the following claims.*

1. *If X is separable then $\overline{X \cap M} = X$.*
2. *If X is Lindelöf and $\mathcal{U} \in M$ is an open cover of X then $M \cap \mathcal{U}$ covers X.*

Exercise 2.2.30. *Suppose that $F : \omega_1 \to [\omega_1]^{<\omega}$. Show that there is a stationary $S \subset \omega_1$ so that $\{F(\xi) : \xi \in S\}$ is a Δ-system.*

[6] In other words, this topology satisfies the countable chain condition.

Exercise 2.2.31. *Show that any family of* $(2^{\aleph_0})^+$*-many countably infinite sets contains a* Δ*-system of the same size.*

Problem 2.2.32. *Show that there is a family of countable sets* $\mathcal{A}$ *of size continuum so that* $\mathcal{A}$ *is totally ordered by the subset relation (i.e., for any* $x \neq y \in A$ *either* $x \subset y$ *or* $y \subset x$*).*

Problem 2.2.33. *Let* k *be finite and suppose that* F *is a family of sets each of finite size* s*. Show that if* $|F| > s!\,(k-1)^s$ *then* F *contains a* Δ*-system with at least* k *elements.*

Problem 2.2.34. *Suppose that* $\mathcal{A}$ *is a family of subsets of* $\mathbb{R}$ *and for any* $a, b \in \mathcal{A}$*,* $a \cap b$ *is finite. Prove that* $\mathcal{A}$ *has size at most continuum.*

Problem 2.2.35. *Prove that there is no strictly increasing sequence* $(F_\xi)_{\xi<\omega_1}$ *of closed subsets of* $\mathbb{R}$*.*

2.2.5 Graphs with uncountable chromatic number

The next group of applications concerns the following question: what subgraphs must appear in a graph with large chromatic number? Is it true that certain cycles, paths or say highly connected graphs must embed into every graph with large enough chromatic number? A seminal result of P. Erdős is that for any finite k and ℓ, there is a graph G of chromatic number k which contains no cycles of length $\leq \ell$ [12]. So the chromatic number can be arbitrary large, while the ℓ-neighbourhood of any vertex (i.e., the other vertices of distance $\leq \ell$) must form a tree (that is, contains no cycles). Now, trees have chromatic number 2; so we see that there are graphs with arbitrary large chromatic number which locally have the smallest possible chromatic number.

Quite interestingly, the above result does not extend to graphs with uncountable chromatic number [10]. In fact, the following theorem holds where $H_{\omega,\omega+1}$ denotes the so-called infinite half-graph i.e., the graph on vertices $\{u_k : k < \omega\} \cup \{v_k : k \leq \omega\}$ and edges $u_k v_\ell$ for $k \leq \ell \leq \omega$. See Figure 2.5 below.

Theorem 2.2.36 (A. Hajnal and P. Komjáth, 1984 [17])**.** *Any graph* G *of uncountable chromatic number must contain the graph* $H_{\omega,\omega+1}$*.*

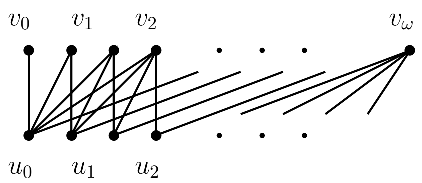

Figure 2.5: The infinite half-graph $H_{\omega,\omega+1}$

In particular, all even cycles and actually all finite bipartite graphs must appear in any graph of uncountable chromatic number. We mention that the lack of finite half-graphs of a given size also has various consequences on the structure and regularity properties of the graph [30].

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Proof. Suppose that G is a counterexample of minimal size κ. Take a sequence of elementary submodels $(M_\alpha)_{\alpha<\mathrm{cf}(\kappa)}$, each of size $<\kappa$ and containing G. Note that $V_\alpha = V \cap M_{\alpha+1} \setminus M_\alpha$ has countable chromatic number by the minimality of G.

Claim 2.2.37. *For any α and $v \in V_\alpha$, $N(v) \cap M_\alpha$ is finite.*

Proof. Take v as in the claim and let $\{u_n : n < \omega\}$ be an infinite subset of $N(v) \cap M_\alpha$. Observe that for any $n < \omega$, $\bigcap_{k<n} N(u_k)$ must be uncountable by Fact 2.2.5; indeed, the latter intersection is an element of M_α that contains $v \notin M_\alpha$. So, we can select pairwise distinct $v_n \in \bigcap_{k<n+1} N(u_k)$ for $n < \omega$. Now, $\{u_k : k < \omega\} \cup \{v, v_k : k < \omega\}$ is a copy of $H_{\omega,\omega+1}$, a contradiction. $\square$

However, the claim implies that G has countable chromatic number since we can glue together proper colourings of $V \cap M_{\alpha+1} \setminus M_\alpha$. Indeed, suppose that for each α, we fixed $g_\alpha : V_\alpha \to \omega$, a proper colouring. We define $g : V \to \omega$ using the g_α functions and a partition of ω into infinite sets $(I_n)_{n\in\omega}$. In fact, we will make sure that if $v \in V_\alpha$ then $g(v) \in I_{g_\alpha(v)}$. Note that this assumption ensures that $g \restriction V_\alpha$ is again a good colouring (no matter how we pick $g(v)$ in $I_{g_\alpha(v)}$).

First, simply let $g(v) = \min I_{g_0(v)}$ for $v \in V \cap M_0$. Next, suppose that $g \restriction V \cap M_\alpha$ is defined already. Now, if $v \in V_\alpha$ then $N(v) \cap M_\alpha$ is finite so we can let

$$g(v) = \min I_{g_\alpha(v)} \setminus g[N(v) \cap M_\alpha].$$

This definition ensures that there is no conflict between the colours on $V \cap M_\alpha$ and V_α. This finishes the construction of a good colouring g and the proof of the theorem is done.

$\boxtimes$

So we see that not just even cycles but all finite bipartite graphs must embed into any graph with uncountable chromatic number. However, finitely many odd cycles can be avoided: the simplest examples that witness this are *the shift graphs* [10]. Take some cardinal κ and natural number n. We define a graph $Sh_n(\kappa)$ on vertices $[\kappa]^n$ and edges $\bar{a}\bar{b}$ where

$$a_0 < a_1 = b_0 < a_2 = b_1 < \dots < a_{n-1} = b_{n-2} < b_{n-1}.$$

Just looking at the $n = 2$ case, one easily sees that $Sh_2(\kappa)$ has no triangles. Moreover, if κ is at least $\mathfrak{c}^+$ then $Sh_2(\kappa)$ has uncountable chromatic number. In general, $Sh_n(\kappa)$ contains no odd cycles of length at most $2n - 1$ and by choosing κ large enough, the chromatic number of $Sh_n(\kappa)$ can be made arbitrary large. Shift graphs are also interesting because the chromatic number of their finite subgraphs grown relatively slowly.[7]

On the other hand, it was also shown that in any graph of uncountable chromatic number, all but finitely many odd cycles must appear.

[7]Recall that any graph with infinite chromatic number must contain finite subgraphs with arbitrary large finite chromatic number. However, in general, there is no bound on how fast these subgraphs should grow in size relative to their chromatic number.

Theorem 2.2.38 (C. Thomassen, 1983 [43])**.** *If G has uncountable chromatic number then G contains odd cycles of all but finitely many lengths.*

Proof. We can assume that G is connected (since it must have a connected component of uncountable chromatic number). Fix a vertex x and partition the vertices V into $(V_m)_{m<\omega}$ so that $v \in V_m$ iff the shortest path from x to v has m edges. Now, some V_m must induce a subgraph with uncountable chromatic number. We will find odd cycles of all length at least $2m$. For any $k < \omega$, we can find a copy H of the complete bipartite graph $K_{k,k}$ in V_m. Let uv be an edge in H and take paths P, P' from x to u and v of length m, respectively.

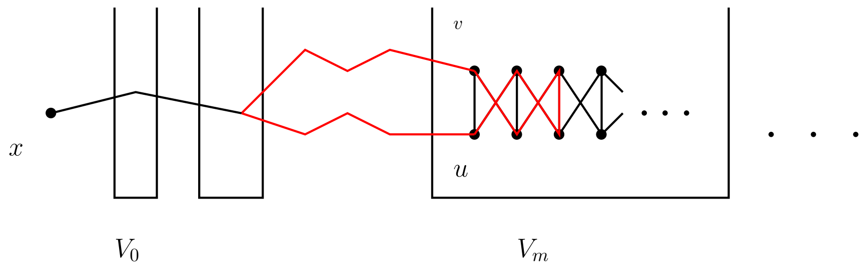


Figure 2.6: Odd cycles in graphs with uncountable chromatic number

Note that $P \cup P'$ has no edges in V_m and contains a path from u to v of even length $\ell \leq 2m$. Moreover, in H, we can connect u to v with odd paths of length $1, 3, 5...$ up to $2k - 1$. So, we get odd cycles of length between $\ell + 1, \ell + 3, \dots, \ell + 2k - 1$. Since we can do this for any $k < \omega$, the proof is done. $\square$

So, the following holds.

Corollary 2.2.39. *A finite graph H embeds into all graphs of uncountable chromatic number if and only if H is bipartite.*

The same question for countable, unavoidable subgraphs H is wide open. Similarly, there is no nice description of the unavoidable finite substructures of uncountably chromatic triple systems [27].

It will be instructive to see how to construct *by hand* sparse graphs with uncountable chromatic number and certain structural properties. Variations on the following construction can produce examples which avoid copies of $H_{\omega,\omega+2}$ [17] or examples without uncountable, infinitely connected subgraphs [41].

A construction scheme for graphs. We will define a graph G on vertex set $V = \omega_1 \times \mathfrak{c}$. We let $\pi : V \to \omega_1$ denote the projection to the first coordinate and let $V_\alpha = \alpha \times \mathfrak{c}$. Nodes with the same projection will never be connected which already implies that $\chi(G) \leq \aleph_1$.

The neighbourhoods of a node (α, ξ) will satisfy that $\pi[N(\alpha, \xi)] \cap \alpha$ is either a finite or type ω cofinal subset of α (i.e., converges to α). This is a natural condition which makes the graph quite sparse.

Now, how do we ensure uncountable chromatic number? The idea is to diagonalise over all possible vertex colourings $g : V \to \omega$ and for each such g, construct a vertex v_g witnessing that the colouring is not good. That is, no matter what is the colour of v_g, there will be another vertex u connected to v_g so that $g(u) = g(v_g)$.

We cannot diagonalise over all colourings $g : V \to \omega$ because there are $2^{|V|}$ many of them while only $|V|$ vertices to choose the witnesses v_g from. Instead, we will look at countable restrictions which reflect important features of the colourings i.e., which colours appear often.

For each $\alpha < \omega_1$ and $\xi < \mathfrak{c}$, we will define $N(\alpha, \xi) \cap V_\alpha$. To do this, list all countable partial functions $g : A \to \omega$ with $A \in [V_\alpha]^\omega$ as $\{g_\xi : \xi < \mathfrak{c}\}$. We use g_ξ to define $N(\alpha, \xi) \cap V_\alpha$. Let

$$I_\xi = \{n \in \omega : \sup \pi[g_\xi^{-1}(n)] = \alpha\},$$

the set of large colours for g_ξ. If I_ξ is empty then we let $N(\alpha, \xi) \cap V_\alpha$ be empty. Otherwise, define $N(\alpha, \xi) \cap V_\alpha = \{u_k : k < \omega\} \subset \operatorname{dom} g_\xi$ so that

1. $\{\pi(u_k) : k < \omega\}$ converges to α, moreover
2. for any $n \in I_\xi$, there is some $k < \omega$ so that $g(u_k) = n$.

This ends the construction of G and we need to show that for any $g : V \to \omega$, there is an edge uv so that $g(u) = g(v)$. Now, take a countable elementary submodel $M \prec H(\mathfrak{c}^+)$ with $G, g \in M$. Let $\alpha = M \cap \omega_1$ and find a ξ so that $g_\xi = g \restriction M \cap V$.

Let $v = (\alpha, \xi)$ and $n = g(v)$. We claim that $n \in I_\xi$. Indeed, note that $\pi[g^{-1}(n)] \in M$ and $\alpha \in \pi[g^{-1}(n)]$ so

$$\sup(\pi[g^{-1}(n)] \cap M) = \sup \pi[g_\xi^{-1}(n)] = \alpha$$

by Fact 2.2.5.

In turn, by the definition of $N(v)$, there is $u \in N(v)$ so that $g(u) = n = g(v)$. This proves that g was not a good colouring. So, $\chi(G) = \aleph_1$ must hold.

A possible improvement of the previous construction is the following. Call a path $P = v_0 v_1 \ldots$ in G monotone if $\pi(v_0), \pi(v_1), \ldots$ is increasing. By a non-trivial modification of the above argument, one can exclude cycles from G which are the union of two monotone paths. This immediately yields that G will omit triangles.

Summary

We saw that elementary submodels provide a unified approach to some important themes:

- we can define *closed substructures* of a given size of arbitrary large structures;
- we can *reflect* properties of the global structures to small substructures which, in turn, support diagonalisation arguments;
- we can *decompose a large object into a continuous sequence* of well-behaving substructures which often serves inductive proofs;
- elementary submodels can also operate as *barriers* in a structure X by separating points of $X \cap M$ from $X \setminus M$.

Exercises and problems

Exercise 2.2.40. *Prove that any monotone $f:\omega_1 \to \mathbb{R}$ is eventually constant.*

Exercise 2.2.41. *Suppose that $X \in H(\theta)$ and $S \subset \kappa$ is stationary for some regular, uncountable κ. Then there is an $M \prec H(\theta)$ so that $X \in M$ and $M \cap \kappa \in S$.*

Exercise 2.2.42. *Suppose that $<^*$ is some well order on ω_1 of type ω_1. Prove that there is a club $C \subset \omega_1$ so that $\alpha < \beta \in C$ implies that $\alpha <^* \beta$ (i.e., the standard well order and $<^*$ must agree on C).*

Exercise 2.2.43. *Suppose that $\mathcal{F}$ is a d-almost disjoint family of countably infinite sets. Find a colouring $c:\bigcup\mathcal{F} \to \omega$ so that c assumes all colours on any element $a \in \mathcal{F}$.*

Exercise 2.2.44. *Take some cardinal κ and natural number n. We define a graph $Sh_n(\kappa)$ on vertices $[\kappa]^n$ and edges $\bar{a}\bar{b}$ where*

$$a_0 < a_1 = b_0 < a_2 = b_1 < \cdots < a_{n-1} = b_{n-2} < b_{n-1}.$$

Show that for any cardinal χ there is a large enough κ, so that $Sh_n(\kappa)$ has chromatic number at least χ.

Problem 2.2.45. *Suppose $S \subset \omega_1$ is a set of limit ordinals and that C_α is a cofinal subset of α of type ω for $\alpha \in S$. Show that S is non-stationary if and only if $\{C_\alpha : \alpha \in S\}$ is essentially disjoint.*

Problem 2.2.46. *Find a 2-almost disjoint family of sets which is not essentially disjoint.*

Problem 2.2.47. *Find a family $\mathcal{F}$ of countably infinite, almost disjoint sets that has uncountable chromatic number i.e., for any $c:\bigcup\mathcal{F} \to \omega$ there is some $a \in \mathcal{F}$ so that $c \restriction a$ is constant.*

The *colouring number* of a graph G, denote by $Col(G)$, is the minimal κ so that $V(G)$ has a well order $<$ such that for any $v \in V$, $\{u < v : uv \in E(G)\}$ has size $< \kappa$.

Problem 2.2.48. *Prove that $\chi(G) \leq Col(G)$ that is, the chromatic number is at most the colouring number.*

Problem 2.2.49. *Find a graph G such that $\chi(G) < Col(G)$.*

⋆

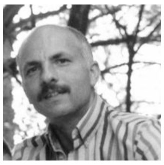
Open Problem 2.2.50. *Is there a graph with uncountable chromatic number that contains no infinitely connected subgraph?*

Open Problem 2.2.51. *Does every two uncountably chromatic graph contain a common subgraph of chromatic number 4?*

Open Problem 2.2.52. *Does every two uncountably chromatic graph contain a triangle-free subgraph of uncountable chromatic number?*

2.3 Balogh's Q-set space

To conclude this chapter, we will see a construction that combines a number of the previous ideas (diagonalisations of length continuum, elementary submodels and Δ-systems) in a more complex topological setting.



Zoltán Balogh 1953 - 2002

Theorem 2.3.1 (Z. Balogh, 1991 [3]). *There is a regular, T_1 topological space X with the following properties:*

(a) any subset Y of X is G_δ, and

(b) X is not σ-discrete i.e., the union of countably many discrete sets.

Such spaces are refereed to as *Q-set spaces*. Why is this theorem interesting? Well, consider the first feature, all subsets being G_δ i.e., the intersection of countably many open sets. This is a certain notion of smallness: for example, any countable topological space satisfies this property. Under various set-theoretic assumptions, like Martin's Axiom, any separable, metric space of size $< \mathfrak{c}$ has this feature. Moreover, any such *Q-set* (i.e., a separable, metrizable Q-set space) must have universal measure 0.[8] On the other hand, it is not hard to see that under CH, there are no Q-sets.

In turn, it is natural to look for other (not necessarily metrizable) examples that could exist without any extra assumptions. However, there is a simple condition on any topological space that makes feature (a) hold trivially: if the space X is the countable union of closed discrete sets, then X has all subsets G_δ.[9] When we exclude this case, we arrive at the right question which turned out to be quite hard to solve.

Proof. We will present a simplified version of Balogh's construction. In fact, we will not bother making the space regular, just T_1. This will ease notation and will let us focus on the more important features of the space.

Our space will have size continuum so we just let $X = \mathfrak{c}$ and define the topology there. We shall start by the cofinite topology τ_0 with base $\mathcal{B}_0 = \{X \setminus F : F \in [X]^{<\omega}\}$. Our final goal is to ensure that all subsets $Y \subset X$ are G_δ and we do this by the most naive approach. By considering each $Y \subset X$ separately, we add a decreasing sequence of sets $(G_{Y,n})_{n<\omega}$ to our basis such that

$$Y = \bigcap_{n<\omega} G_{Y,n}.$$

The final topology τ is generated by $\mathcal{B} = \mathcal{B}_0 \cup \{G_{Y,n} : Y \in \mathcal{P}(X), n < \omega\}$ as a subbasis (see below).

This is all good so far, but we need to be careful not to make the space σ-discrete. Let's understand how basic neighbourhoods will look like in the final topology: these are simply finite intersections from $\mathcal{B}$ which can be characterised by a cofinite set and a finite partial function from $\mathcal{P}(X)$ to ω (the set of the latter maps will be denoted by $Fn(\mathcal{P}(X), \omega)$). Now, given some $U \in Fn(\mathcal{P}(X), \omega)$, we let

$$[U] := \bigcap_{(Y,n) \in U} G_{Y,n}.$$

[8] I.e., Q-sets are null with respect to any regular Borel measure on $\mathbb{R}$.

[9] Indeed, if $D \subset X$ is closed and discrete then any subset Y of D is closed in X. So, if X is $\sigma-$closed discrete then any subset Y is the countable union of closed sets. In turn, any subset Y must be G_δ as well.

So a basic neighbourhood of a point x will look like $x \in [U(x)] \setminus F(x)$ where $F(x)$ is a finite subset of X and $U(x) \in Fn(\mathcal{P}(X), \omega)$.

In turn, we need the following property:

> if $X = \bigcup_{k\in\omega} X_k$ and the maps $U_k : X_k \to Fn(\mathcal{P}(X), \omega)$ and $F_k : X_k \to [X]^{<\omega}$ code a neighbourhood assignment then there is an $n < \omega$ and $x \neq y \in X_k$ so that $y \in [U_k(x)]$.

In other words, U_k, F_k does not witness that X_k is discrete. This feels quite similar to the chromatic number problems we considered before. Again, on the face of it, we need to find witnesses for $2^{\mathfrak{c}}$ many possible partitions from a collection of only $\mathfrak{c}$ points. The trick will be again to consider the trace of these partitions and neighbourhood assignments on nice, countable sets.

We say that (A, u) is a *control pair* if A is a countable subset of X and

$$u : A \to Fn(\mathcal{P}(A), \omega)$$

so that $\alpha < \alpha' \in A$ implies $\operatorname{dom} u(\alpha) \cap \operatorname{dom} u(\alpha') = \emptyset$. So u looks like a neighbourhood assignment with some additional property on the domains. Note that there are only $\mathfrak{c}$ many control pairs, so we can list them as $(A_\beta, u_\beta)_{\beta<\mathfrak{c}}$. We also arrange that $A_\beta \subset \beta$ for any $\beta < \mathfrak{c}$ and so $\beta \notin A_\beta$.

Now, let's describe the construction: given $Y \subset X$, we define $G_{Y,n}$ for $n < \omega$ as follows. For each $\beta \in X = \mathfrak{c}$, we decide if $\beta \in G_{Y,n}$ or $\beta \notin G_{Y,n}$: we put $\beta \in G_{Y,n}$ if and only if one of the following conditions hold:

1. $\beta \in Y$, or
2. there is an $\alpha \in A_\beta$ so that $Y \cap A_\beta \in \operatorname{dom} u_\beta(\alpha)$ and $n \leq \max u_\beta(\alpha)$.

Note that the second condition can hold for at most one $\alpha \in A_\beta$ which uniquely determines those finitely many n's such that $\beta \in G_{Y,n}$. This ensure that if $\beta \in G_{Y,n}$ and $m < n$ then $\beta \in G_{Y,m}$ i.e., the sequence is decreasing. Furthermore, $Y = \bigcap_{n<\omega} G_{Y,n}$ as desired. This ends the construction of the topology.

We need to prove that X is not σ-discrete in the final topology. Before that, let us point out the intuition behind the above definition: think of $u = u_\beta$ as a neighbourhood assignment. Then the second condition essentially says that whenever $(Y, n) \in u(\alpha)$, we put β into $G_{Y,n}$ as well. In other words, we work to ensure $\beta \in [u(\alpha)]$ so the neighbourhood assignment u cannot witness that α and β are in the same discrete set.

Now, assume that $X = \bigcup_{n\in\omega} X_k$ and $U_k : X \to Fn(\mathcal{P}(X), \omega)$ together with $F_k : X_k \to [X]^{<\omega}$ codes a neighbourhood assignment. We need some $n < \omega$ and $\alpha < \beta \in X_k$ so that

$$\beta \in [U_k(\alpha)] \setminus F_k(\alpha).$$

Take a countable $M \prec H(\theta)$ so that

$$\tau, (X_k)_{n<\omega}, (U_k)_{n\in\omega}, (F_k)_{n\in\omega}, \mathcal{B}, \ldots \in M.$$

We let $A = M \cap \mathfrak{c}$. Now, we define $u : A \to Fn(\mathcal{P}(A), \omega)$ so that $\{\operatorname{dom} u(\alpha) : \alpha \in A\}$ is pairwise disjoint and if $v \in M$ such that v is an infinite partial function from $\mathfrak{c}$ to $Fn(\mathcal{P}(X), \omega)$ with $\{\operatorname{dom} v(\alpha) : \alpha \in \operatorname{dom} v\}$ pairwise disjoint then there is an $\alpha \in \operatorname{dom} v \cap \operatorname{dom} u$ so that

$$u(\alpha) = \{(Y \cap A, n) : (Y, n) \in v(\alpha)\}.$$

In some sense, u diagonalises all partial neighbourhood assignments from M. Why is this possible? List all $v \in M$ that we need to consider as $(v_n)_{n\in\omega}$. Select $\alpha_n \in \operatorname{dom} v_n$ so that $\{\operatorname{dom} v_n(\alpha_n) : n < \omega\}$ is pairwise disjoint. Now, let

$$u(\alpha) = \begin{cases} \{(Y \cap A, n) : (Y, n) \in v_n(\alpha_n)\} & \text{if } \alpha = \alpha_n, \\ \emptyset & \text{otherwise.} \end{cases}$$

We shall prove that (A, u) is a control pair i.e., that $\operatorname{dom} u(\alpha_n)$ and $\operatorname{dom} u(\alpha_m)$ are disjoint for $n < m < \omega$. That is, we need that $Y \cap A \neq Y' \cap A$ for $Y \in \operatorname{dom} v_n(\alpha_n)$ and $Y' \in \operatorname{dom} v_n(\alpha_n)$. Note that $Y \neq Y'$ and also, $Y, Y' \in M$. So this must be witnessed in M i.e.,

$$Y \cap A = Y \cap M \neq Y' \cap M = Y' \cap A$$

as desired.

In turn, the control pair (A, u) was enumerated at some stage $\beta < \mathfrak{c}$ as (A_β, u_β). Fix $n < \omega$ so that $\beta \in X_k$. Our goal is to find an $\alpha \in A \cap X_k$ so that $\beta \in [U_k(\alpha)] \setminus F_k(\alpha)$. Note that it suffices to arrange $\beta \in [U_k(\alpha)]$ since $\alpha \in A \subset M$ implies that $F_k(\alpha) \subset M$ but $\beta \notin M$.

Look at $U_k(\beta)$ and recall that

$$\beta \in [U_k(\beta)] = \bigcap_{(Y,n)\in U_k(\beta)} G_{Y,n}.$$

Let $W := M \cap U_k(\beta)$ which is an element of M (being a finite subset of M). The set

$$E = \{\alpha \in X_k : W \subset U_k(\alpha)\}$$

is also an element of M (since $X_k, U_k \in M$) and $\beta \in E \setminus M$ so E must be uncountable. Pick, in M, a maximal $E_0 \subset E$ so that $\{U_k(\alpha) \setminus W : \alpha \in E_0\}$ has pairwise disjoint domains. Note that E_0 must be infinite, otherwise we could add β which contradicts its maximality (just as in the proof of the Δ-system lemma).

Now, consider $v : E_0 \to Fn(\mathcal{P}(X), \omega)$ defined by $v(\alpha) = U_k(\alpha) \setminus W$; this function v is in M. So, by the definition of our control pair and the map $u = u_\beta$, there is some α so that

$$\begin{aligned} u_\beta(\alpha) &= \{(Y \cap A_\beta, n) : (Y, n) \in v(\alpha)\} \\ &= \{(Y \cap A_\beta, n) : (Y, n) \in U_k(\alpha) \setminus W\}. \end{aligned}$$

Our goal is to show that

$$\beta \in [U_k(\alpha)] = \bigcap_{(Y,n)\in U_k(\alpha)} G_{Y,n}.$$

Now, take some $(Y, n) \in U_k(\alpha)$. If $(Y, n) \in W$ then $(Y, n) \in U_k(\beta)$ so $\beta \in G_{Y,n}$. Assume that $(Y, n) \in U_k(\alpha) \setminus W$. Then $(Y \cap A_\beta, n) \in u_\beta(\alpha)$ and so the second clause in the definition of $G_{Y,n}$ takes effect and we put β in $G_{Y,n}$.

This finishes the proof that X is not σ-discrete.

$\boxtimes$

Balogh later improved the above result [4] and constructed a Q-set space that is (hereditarily) paracompact and also perfectly normal.[10]

We should mention that Balogh has another striking application of elementary submodels and ingenious diagonalization. A *Dowker space* is a T_1, normal topological space X (i.e., any two disjoint closed sets can be separated by disjoint open sets) so that $X \times [0,1]$ is not normal any more. There are several constructions of Dowker spaces using various set theoretic assumptions (such as CH). In a breakthrough result, Mary Ellen Rudin proved that such spaces must always exist in ZFC. Her example has size $\aleph_\omega^{\aleph_0}$ and the quest to find small Dowker spaces in ZFC turned out to be one of the hardest problems in general/set-theoretic topology. Then, Balogh constructed a Dowker space of size continuum in ZFC and just as the Q-set space, his example has weight and character $2^{\mathfrak{c}}$. Without going into too much detail, he used the Q-set space idea to ensure normality: the basis for the space is now constructed inductively and as certain pairs of disjoint sets become closed during the induction, he introduces disjoint open sets that separate them. It involves more topology to see how the Dowker property is ensured which is not in the scope of these lectures.

Mary Ellen Rudin 1924 - 2013

Balogh credited Mary Ellen Rudin and her work as inspiration for his construction of the Q-set and Dowker spaces. Still, it is one of the great unsolved problems of point-set topology if there are Dowker spaces of size, weight or character ω_1 in ZFC.

Summary

There are a few important points to take away from Balogh's proof:

- we can often capture the essential information of a large object (such as a partition or neighbourhood assignment) by countable traces;
- the latter supports inductive diagonalisations of length continuum;
- by using elementary submodels, we can reflect whatever finite information we have from the original structure to the countable trace;
- finally, keep in mind that a simple and naive approach can still solve major open problems with enough perseverance.

Exercises and problems

Exercise 2.3.2. *Show that any graph G of uncountable chromatic number contains a copy of K_{n,ω_1} for any $n < \omega$. The latter is the complete bipartite graph with one finite class of size n and another uncountable class.*

Exercise 2.3.3. *Fix a natural number k. Prove that any graph G of uncountable chromatic number contains a k-connected subgraph H i.e., H remains connected after the removal of $< k$ vertices.*

[10] This makes his example almost as nice as a metrizable space.

Exercise 2.3.4. *Prove that the rational distance graph on the plane has no copies of* K_{2,ω_1} *and so it must have countable chromatic number.*

Exercise 2.3.5. *Suppose that* (X, τ) *is a topological space with a point-countable base* $\mathcal{B}$*. That is, for any* $x \in X$, $\{U \in \mathcal{B} : x \in U\}$ *is countable. Let* $(X, \tau), \mathcal{B} \in M \prec H(\theta)$*. Prove that for any* $y \in \overline{X \cap M}$, $\mathcal{B} \cap M$ *contains a neighbourhood base for* y*.*

Exercise 2.3.6. *Suppose that* X *is a separable metric space and* $Y \subset X$ *is* σ*-discrete. Prove that* Y *is countable.*

Exercise 2.3.7. *Suppose that* $2^{\aleph_0} = \aleph_1$ *and* X *is an uncountable, separable metric space. Prove that* X *has a subset* Y *that is not* G_δ*.*

Problem 2.3.8. *Show that if a topological space* X *is* σ*-discrete and any subset is a* G_δ *then actually* X *is* σ*-closed discrete.*

Problem 2.3.9. *Suppose that some* $X \subset 2^{\omega_1}$ *satisfies the following: for any uncountable family* $\{s_\xi : \xi < \omega_1\}$ *of finite functions with pairwise disjoint domain there is a countable* I *so that* $X \setminus \bigcup_{\xi \in I} [s_\xi]$ *is countable.*[11] *Prove that any open cover of* X *has a countable subcover i.e., that* X *is Lindelöf.*

Problem 2.3.10. *Show that for any countable edge-colouring of the complete graph on* ω_2*, one can find an infinite monochromatic path.*

Problem 2.3.11. *Describe an explicit construction of finite triangle-free graphs with arbitrary large finite chromatic number.*[12]

Challenge 2.3.12. *Show that there is a countable subspace* $X \subset 2^{\mathfrak{c}}$ *which has no isolated points and any two non-empty dense subsets of* X *have non-empty intersection.*[13]

⋆

Open Problem 2.3.13. *Is there a 'small' Dowker space i.e., one of size, character or weight* ω_1*?*

Further reading

Introductions to elementary submodels [21, 28]; elementary submodels in topology [8, 15]; introduction to elementary submodels and graph theory [42]; about uncountable graphs and chromatic number [26, 25].

[11] The notation $[s]$ stands for $\{f \in 2^{\omega_1} : s \subset f\}$.

[12] Hint: using a triangle-free graph H, try to build a larger G which is still triangle-free but has bigger chromatic number.

[13] Hint: construct $\{x_n : n < \omega\} \subset 2^{\mathfrak{c}}$ by defining $\{x_n \restriction \alpha : n < \omega\}$ by an induction on $\alpha < \mathfrak{c}$. What should we diagonalise in the construction?

Chapter 3

Coherent maps and minimal walks

3.1 The first uncountable ordinal

We will now look more closely at combinatorial properties of ω_1, the smallest uncountable cardinal. The ordinal ω_1 is an incredibly interesting object witnessing various paradoxical properties. Just consider the following

- The $\in$ relation well-orders ω_1 so that any proper initial segment is countable while the whole order is uncountable.
- For any limit $\alpha \in \omega_1$, there is countable, increasing, type ω sequence cofinal in α but any countable subset of ω_1 is bounded.
- Any proper initial segment of ω_1 has an order preserving/topological embedding into $\mathbb{R}$ but not ω_1 itself.
- Any proper initial segment of ω_1 is metrizable but not ω_1 itself.

This tension between the local and global properties of ω_1 can be used to construct a great variety of interesting mathematical objects.

Since each $\alpha < \omega_1$ is countable, there is an $e_\alpha : \alpha \to \omega$ that is injective (or bijective even, if you wish). Now, if $\alpha < \beta$ and the set $\omega \setminus \operatorname{ran} e_\alpha$ is infinite then we can extend e_α to an injective $e_\beta : \beta \to \omega$ (and still avoid infinitely many values if we wish so). How long can we keep doing this? Since there is no injective $e : \omega_1 \to \omega$, we must fail at some limit step of any construction like that.

However, the following is possible. Call a collection of maps $\mathcal{E}$ *coherent* if for any $e, e' \in \mathcal{E}$, the set

$$\{\xi \in \operatorname{dom} e \cap \operatorname{dom} e' : e(\xi) \neq e'(\xi)\}$$

is finite.

Theorem 3.1.1. *There is a coherent sequence of one-to-one maps $e_\alpha : \alpha \to \omega$ for $\alpha < \omega_1$.*

Proof. We construct $(e_\alpha)_{\alpha<\omega_1}$ by induction on $\alpha < \omega_1$ so that

1. $e_\alpha : \alpha \to \omega$ is injective,
2. $\operatorname{ran}(e_\alpha)$ is co-infinite,
3. for any $\beta < \alpha$, $\{\xi < \beta : e_\beta(\xi) \neq e_\alpha(\xi)\}$ is finite.

At successor steps, we simply let $e_{\alpha+1} = e_\alpha \cup \{(\alpha, m)\}$ so that $m \notin \operatorname{ran} e_\alpha$.

In limit steps, we select a strictly increasing cofinal sequence $(\alpha_n)_{n<\omega}$ is α. Naively, we might just take $g_n = e_{\alpha_n} \restriction \alpha_n \setminus \alpha_{n-1}$ and look at $\bigcup\{g_n : n \in \omega\}$ as a candidate for e_α but this can fail to be 1-1 or fail the requirement on the range. So, we do the following correction: first, let $g_0 = \bar{g}_0$. Next, we modify g_1 at finitely many points to get $\bar{g}_1$ so that $\bar{g}_0 \cup \bar{g}_1$ is 1-1 (this is possible, there could be only finitely many values $k < \omega$ that are assumed by both g_0 and g_1). Furthermore, we reserve a value k_1 outside the range of $\bar{g}_0 \cup \bar{g}_1$ and promise to keep k_1 out of the range of all the $\bar{g}_n$. We proceed similarly: at stage $n < \omega$, we modify g_n at finitely many points to get $\bar{g}_n$ so that $\cup\{\bar{g}_i : i \leq n\}$ is 1-1 and does not assume the values $k_1, k_2 \ldots k_{n-1}$. Finally, we reserve a new value k_n to be avoided.

The function e_α we constructed is 1-1 and has co-infinite range. Moreover, note that $e_{\alpha_n} =^* \bigcup\{\bar{g}_l : l \leq n\}$ where $=^*$ denotes equality modulo a finite set. So, if $\beta < \alpha$ and we pick n so that $\beta < \alpha_n$ then

$$e_\beta =^* e_{\alpha_n} \restriction \beta =^* \bigcup\{\bar{g}_l : l \leq n\} \restriction \beta = e_\alpha \restriction \beta.$$

This ends the proof. ⊠

The coherent, 1-1 sequence $(e_\alpha)_{\alpha<\omega_1}$ gives rise to a so-called *Aronszajn tree*. A set theoretic tree is a partially ordered set $(T, <)$ so that for any $t \in T$, $t^\downarrow = \{s \in T : s < t\}$ is well ordered. For example, the set $2^{<\omega}$ of finite sequences of natural numbers forms a tree with end-extensions. Any tree admits a *height* function: the height of t is simply the order type of $t^\downarrow$. The height of the tree itself is the supremum of all the heights of elements of T.

Now, an $\aleph_1$-*tree* is a tree of height ω_1 with all levels countable. For example, $(\omega_1, \in)$ is an $\aleph_1$-tree. Aronszajn-trees form the other extreme: we say that T is Aronszajn if T is an $\aleph_1$-tree without uncountable chains.

Going back to coherent, 1-1 sequences, we can form

$$T = \{e_\beta \restriction \alpha : \alpha \leq \beta < \omega_1\}.$$

Now, $(T, \subseteq)$ is a downward closed, uncountable subtree of $\omega^{<\omega_1}$ which has countable levels but no uncountable chains i.e., T is Aronszajn. The existence of such trees show that König's well-known theorem[1] does not extend to ω_1.

Another important note is that a coherent sequence of 1-1 (or even finite-to-one) maps $e_\alpha : \alpha \to \omega$ gives rise to finite sets

$$F_n(\alpha) = \{\xi \leq \alpha : \xi = \alpha \text{ or } e_\alpha(\xi) \leq n\}.$$

Note that $\alpha + 1 = \bigcup\{F_n(\alpha) : n < \omega\}$ and for any $\alpha < \beta$ and large enough $n < \omega$, $F_n(\alpha) \subseteq F_n(\beta)$. That is, we can coherently decompose the countable ordinals into increasing sequences of finite sets. Such decompositions will play an important role later.

[1] Recall that König's theorem says that any infinite tree with finite levels must have an infinite branch

3.2 Some Ramsey theory on ω_1

Ramsey theory is concerned with the phenomena that any *large enough* object, no matter how random it might look, necessarily contains regular substructures of a given size. A popular way to phrase this is that 'complete disorder is impossible.' There are various branches of Ramsey theory that concern finding regularity in geometric objects, general topological space, Banach spaces, and various fascinating number and graph theoretic aspects. We will look at the most fundamental case: partitions of $[X]^n$ for an unstructured infinite set X.

One might consider the pigeon hole principle the simplest, 1-dimensional Ramsey-type statement: any infinite set of size κ, when partitioned into finitely many pieces, contains a monochromatic piece of size κ. The classical theorem of F. P. Ramsey from 1930 [35] captures the multi-dimensional analogue: for any finite n and k,

$$\omega \to (\omega)^n_k$$

i.e., for any colouring $c : [\omega]^n \to k$ there is an infinite $A \subset \omega$ so that $c \restriction [A]^n$ is constant.[2]

On the other hand, Sierpinski showed in 1933 [39] that for any $\lambda \leq \mathfrak{c}$,

$$\lambda \not\to (\omega_1)^2_2$$

i.e., there is a colouring $c : [\lambda]^2 \to 2$ so that both colours appear on any uncountable subset of λ. The colouring c is fairly easy to define: take any well-ordered set of λ-many reals $(r_\alpha)_{\alpha<\lambda}$ and for $\alpha < \beta < \lambda$, let $c(\alpha, \beta) = 0$ if and only if $r_\alpha <_{\mathbb{R}} r_\beta$. That is you compare the Euclidean ordering and the well-order. The fact that these two orders have no common uncountable suborder ensures that c witnesses Sierpinski's relation.

What if we use more than 2 colours? First, there is a very strong limitation to prove anything about ω-colourings: consider the map $\Delta : [2^\omega]^2 \to \omega$ defined by

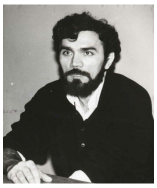
$$\Delta(f, g) = \min\{n < \omega : f(n) \neq g(n)\}.$$

Note that there are no monochromatic triangles for Δ so, using the arrow notation, we have

$$\mathfrak{c} \not\to (3)^2_\omega.$$

The same negative relation holds for any $\lambda \leq \mathfrak{c}$, in particular for ω_1 (indeed, just restrict Δ to an appropriate set of reals).

How about finite colourings? It was open for a long time if one can at least *reduce the number of colours* or omit a colour on some uncountable subset. This question was eventually settled by Todorcevic.



Stevo Todorcevic 1955 -

Theorem 3.2.1 (Todorcevic, 1987 [45]). *There is a colouring $c : [\omega_1]^2 \to \omega_1$ so that for any uncountable $X \subset \omega_1$, $c[X]^2 = \omega_1$.*

There are many variations of this result based on *minimal walks* (due to Todorcevic and Moore, yielding colourings with stronger combinatorial properties), and different proofs using the complete binary tree or special Aronszajn trees; all of this is covered in Todorcevic's

[2] This is the so-called arrow notation for partition relations which was introduced by Erdős and Rado.

book on minimal walks [47]. We will present a simple argument using an Aronszajn-tree and elementary submodels (the argument below is a mixture of proofs due to Todorcevic and Velleman).

We will make use of the following fact which can be thought of as trimming the Aronszajn tree.

Fact 3.2.2. *Suppose that T is an Aronszajn tree. Then T has a subtree $\hat{T}$ so that any $t \in \hat{T}$ has uncountably many extensions in $\hat{T}$.*

Now we are ready to prove the theorem.

Proof. [47, Lemma 5.4.1] The proof consists of two parts: first, a highly non-trivial definition to get a colouring $c : [\omega_1]^2 \to \omega_1$ so that for any uncountable $X \subset \omega_1$, $c[X]^2$ contains a club. Given such a c, the second and much easier part is that we can find an $f : \omega_1 \to \omega_1$ so that $f \circ c$ witnesses the theorem. Indeed, we can take any f so that $f^{-1}(\xi)$ is stationary for all $\xi < \omega_1$.

Let us work on finding c now. Suppose that T is an Aronszajn tree and let us assume that any two nodes $s, t \in T$ have a greatest lower bound $s \wedge t$ (note that $s \wedge t = s$ if $s \leq t$). For each $t \in T$, we define $F_n(t)$ for $n < \omega$ so that

1. $F_n(t) \subset t^{\downarrow} \cup \{t\}$ is a finite set with maximal element t, and
2. $\bigcup_{n<\omega} F_n(t) = t^{\downarrow} \cup \{t\}$.

This is of course possible since each $t^{\downarrow}$ is countable. For $s, t \in T$ with $ht(s) \leq ht(t)$, we define $m = \min\{n < \omega : s \wedge t \in F_n(t)\}$ and now let

$$[s, t] = \min\{w \in F_m(t) : ht(w) \geq ht(s)\} \in T.$$

We will now show that for any uncountable $X \subset T$ and countable $M \prec H(\theta)$ with $X, T \in M$, there is $s, t \in T$ so that

$$M \cap \omega_1 = ht([s, t]).$$

So the map $c : [T]^2 \to \omega_1$ defined by $c(s, t) = ht([s, t])$ will satisfy that $c[X]^2$ contains a club for any uncountable $X \subset T$.[3]

Pick any $t \in X \setminus M$ and let $v \leq t$ so that $ht(v) = M \cap \omega_1$. Our goal is to find $s \in X \cap M$ so that $[s, t] = v$.

Find an m_0 so that $v \in F_{m_0}(t)$ and let u_0 be the maximum of $v^{\downarrow} \cap F_{m_0}(t)$.

Claim 3.2.3. *There is some $u < v$ above u_0 and uncountable $Y \subset X$ in M so that for any $s \in Y$, $s \wedge t = u$.*

Proof. Note that $Y_0 = \{t' \in X : t' > u_0\} \in M$ and $t \in Y_0$ so Y_0 must be uncountable. Let Y_1 be a subtree of Y_0 in M in which any node has uncountably many extensions. Now, Y_1 cannot be a chain so we can find two incomparable elements in $Y_1 \cap M$ one of which is not below t. Fix $y_1 \in Y_1 \cap M$ to be such a point not in $t^{\downarrow}$ and let $Y = \{s \in Y_1 : s > y_1\}$. Then Y is an uncountable subset of X and $s \wedge t = y_1 \wedge t$ for any $s \in Y$. That is $u = y_1 \wedge t$ satisfies the requirements of the claim. □

[3]It does not matter that c is defined on pairs of T instead of pairs of ω_1. Any such map can be transformed to a colouring $c' : [\omega_1]^2 \to \omega_1$ by an arbitrary bijection $\omega_1 \to T$.

Now find the minimal m so that $u \in F_m(t)$ and note that $m > m_0$ as $u > u_0$. In turn, $v \in F_m(t)$. Now, find $s \in Y \cap M$ so that $ht(s) > ht(v')$ for any $v' \in F_m(t) \cap M$. Recall that $u = s \wedge t$ so

$$[s,t] = \min\{w \in F_m(t) : ht(w) \geq ht(s)\} = v$$

as desired.

$\boxtimes$

However, there are some positive Ramsey results that one can prove. The most important ones are summarized below.

1. (Erdős-Dushnik-Miller, 1941) For any infinite κ,

$$\kappa \to (\kappa, \omega+1)^2_2$$

i.e., for any colouring $c : [\kappa]^2 \to 2$ either there is an $A \subset \kappa$ of size κ so that $c \restriction [A]^2$ is constant 0 or there is an $A \subset \kappa$ of order type $\omega+1$ so that $c \restriction [A]^2$ is constant 1.

2. (Erdős-Rado, 1956 [13]) For any infinite κ,

$$(2^\kappa)^+ \to (\kappa^+)^2_\kappa$$

i.e., for any colouring $c : [(2^\kappa)^+]^2 \to \kappa$ there is an $A \subset \kappa$ of size κ^+ so that $c \restriction [A]^2$ is constant.

The last theorem has an analogue for colouring n-tuples but one needs to iterate the exponential function $n-1$ times:

$$(\beth_{n-1}(\kappa))^+ \to (\kappa^+)^n_\kappa.$$

The *beth-function* is defined by $\beth_1(\kappa) = \kappa$ and $\beth_{n+1}(\kappa) = 2^{\beth_n(\kappa)}$.

Complementing Todorcevic and Sierpinski's result, Shelah proved the following:

3. (Shelah, 1988 [37]) Consistently (modulo some large cardinals),

$$\mathfrak{c} \to [\omega_1]^2_{<\omega,3}$$

i.e., for any colouring $c : [\mathfrak{c}]^2 \to r$ with $r < \omega$ there is an uncountable $X \subset \mathfrak{c}$ so that $c \restriction [X]^2$ assumes at most 2 colours.

It is still open if the above positive relation is consistent with small values of $\mathfrak{c}$ i.e., with $\mathfrak{c} < \aleph_\omega$.

Exercises and problems

The ordinal ω_1 has a natural order topology. Basic open neighbourhoods of β are the half-open intervals $(\alpha, \beta]$ where $\alpha < \beta$. So the isolated points are exactly the successor ordinals.

Exercise 3.2.4. *Prove that any continuous $f : \omega_1 \to \mathbb{R}$ is eventually constant.*

Exercise 3.2.5. *Prove that any continuous $f : \mathbb{R} \to \omega_1$ has countable range.*

Exercise 3.2.6. *Show that any graph G of chromatic number at least $\mathfrak{c}^+$ must contain a copy of $H_{\omega,\mathfrak{c}^+}$.*

Exercise 3.2.7. *Prove Kőnig's theorem: any finitely branching, infinite tree must contain an infinite branch.*

Exercise 3.2.8. *Let $(e_\alpha)_{\alpha<\omega_1}$ be a coherent, 1-1 sequence and let $T = \{f \in \omega^{<\omega_1} : (\exists \alpha \in \omega_1) f =^* e_\alpha\}$. Prove that T is an Aronszajn-tree.*

Exercise 3.2.9. *Let $T \subset \omega^{<\omega_1}$ be an Aronszajn tree. Define a relation $<_\ell$ on T so that $s <_\ell t$ if $s(\xi) \supseteq t(\xi)$ or $s(\xi) < t(\xi)$ for the minimal ξ so that $s(\xi) \neq t(\xi)$. Prove that $<_\ell$ is a linear order.*

Problem 3.2.10. *Let T be the tree consisting of all closed subsets of a stationary set S. Show that T cannot be partitioned into countably many antichains.*

Problem 3.2.11. *Suppose that G is a graph on ω_1 and for any limit α, $\alpha \cap N(\alpha)$ is closed and discrete in the order topology of α. Prove that given any pairwise disjoint, uncountable family F of finite subsets of ω_1, there is $a \neq b \in F$ so that there are no edges between a and b. (This statement implies that under $MA_{\aleph_1}$, such graphs must have countable chromatic number.)*

Problem 3.2.12. *Let T be an Aronszajn subtree of $\omega^{<\omega_1}$. Show that there is no strictly increasing map from $(T, <_\ell)$ to $\mathbb{R}$ (see Exercise 3.2.9 for the definition of $<_\ell$).*

Problem 3.2.13. *Let $V = \{f : \alpha \to \omega$ injective, $\alpha < \omega_1\}$ and $fg \in E$ if $f \subset g$ or $g \subset f$. Prove that $G = (V, E)$ is uncountably chromatic.*

3.3 Walks on ordinals

In this section, we shall analyse *minimal walks along C-sequences*. A C-sequence is essentially a ladder system extended to all of ω_1 in a very natural way. That is, for the rest of this chapter, we shall fix $(C_\alpha)_{\alpha<\omega_1}$ so that

1. $C_{\alpha+1} = \{\alpha\}$, and
2. C_α is a cofinal, type ω subset of α if α is limit.

In certain situations, it is useful to assume that if α is limit then C_α consists of successor ordinals only.

Now, each such C-sequence defines a graph on ω_1 by declaring $\alpha\beta$ an edge if $\alpha \in C_\beta$. We are interested in monotone decreasing walks in this graph which greedily step towards a fixed destination. For $\alpha < \beta$, we define *the step from β to α* by

$$\text{step}(\alpha, \beta) = \min C_\beta \setminus \alpha.$$

That is, we find the minimal neighbour of β that is still at least α.

The *minimal walk from β to α* is the iteration of the step function until we reach α. More precisely, we let $\beta_0 = \beta$, $\beta_1 = \text{step}(\alpha, \beta_0)$ and if $\beta_n \neq \alpha$ then

$$\beta_{n+1} = \text{step}(\alpha, \beta_n).$$

Note that $\beta_0 > \beta_1 > \ldots \geq \alpha$ so this process must terminate in finitely many steps by reaching α. If β_n is a successor ordinal and above α then we step down to its predecessor i.e., $\beta_{n+1} = \beta_n - 1$. If β_n is limit, we let β_{n+1} be the first element of the cofinal sequence C_{β_n} that is still at least α.

The (finite) collection of the nodes that appear in the walk is called the *trace* and is denote by

$$Tr(\alpha, \beta) = \{\beta_0, \beta_1, \ldots, \beta_n\}.$$

Just as the minimal walk itself, the trace has a concise recursive definition by

$$\text{Tr}(\alpha, \beta) = \{\beta\} \cup \text{Tr}(\alpha, \text{step}(\alpha, \beta))$$

with boundary value $\text{Tr}(\alpha, \alpha) = \{\alpha\}$.

So we have defined a two-place set-mapping

$$\text{Tr} : [\omega_1]^2 \to [\omega_1]^{<\omega}.$$

Without any further preparation, we can prove a crucial fact about the expansion of the trace function on uncountable sets.

Lemma 3.3.1 (Expansion lemma)**.** *Suppose that $X \subset \omega_1$ is uncountable. Then the set $\text{Tr}[X]^2 = \bigcup\{\text{Tr}(\alpha, \beta) : \alpha < \beta \in X\}$ contains a club.*

Proof. In fact, we shall prove that $\text{acc}(X)$, the set of accumulation points of X, is a subset of $\text{Tr}[X]^2$. Fix some $\delta \in \text{acc}(X)$. Since X is uncountable, we can find $\beta \in X \setminus \delta$. Our goal is to pick $\alpha \in \delta \cap X$ so that $\delta \in \text{Tr}(\alpha, \beta)$ i.e., the walk from β to α goes through δ.

The idea is the following: by definition, $\delta \in \text{Tr}(\delta, \beta)$ so we would like to choose α in such a way that the walk from β to α starts with the walk from β to δ. Look at the first step

$\mathrm{step}(\delta, \beta) = \min C_\beta \setminus \delta$. If α is above the finite set $C_\beta \cap \delta$ then $\mathrm{step}(\alpha, \beta) = \mathrm{step}(\delta, \beta)$. So the two walks take the same first step to some β_1. Note that such a choice of α is possible since $\sup X \cap \delta = \delta$. Actually, all large enough α below δ satisfies this property.

If $\beta_1 = \delta$ then we are done. Otherwise, repeat the procedure: take α that is larger than both $C_{\beta_0} \cap \delta$ and $C_{\beta_1} \cap \delta$. Now the walk from β to δ and β to α share the first two steps. If $\beta_2 = \delta$ then we are done, otherwise, keep repeating. Note that this process again terminates in at most $|\mathrm{Tr}(\delta, \beta)|$-many steps.

□

Corollary 3.3.2. *There is a set-mapping $c : [\omega_1]^2 \to [\omega_1]^{<\omega}$ so that $c[X]^2 = \bigcup\{c(\alpha, \beta) : \alpha < \beta \in X\}$ is all of ω_1 for any uncountable $X \subseteq \omega_1$.*

Proof. Indeed, take a partition of ω_1 into stationary sets $(S_\xi)_{\xi<\omega_1}$ and let $c(\alpha, \beta) = \{\xi < \omega_1 : Tr(\alpha, \beta) \cap S_\xi \neq \emptyset\}$. □

This corollary almost shows the negative partition relation we proved in Theorem 3.2.1. In fact, with some effort, one can find a map $t : [\omega_1]^2 \to \omega_1$ so that $t(\alpha, \beta) \in \mathrm{Tr}(\alpha, \beta)$ and still, for any uncountable X, $t[X]^2$ contains a club. So, combined with the stationary partition trick, t yields another witness for $\omega_1 \not\to [\omega_1]^2_{\omega_1}$. Defining this function t is not in the scope of this course but the proof can be found in [47, Lemma 5.1.4].

There are several directions one can go from here:

1. systematically analyse the basic characteristics of walks (such as the maximal weight ρ_1, the number of steps ρ_2 or the last step ρ_3);
2. to study more complex colourings using minimal walks and oscillations;
3. apply these colourings in various settings (productivity of the ccc, Banach spaces with few operators, S-and L-spaces);
4. define ρ-functions abstractly and derive a metric theory on ω_1;
5. to derive canonical trees and linear orders from functions associated to walks;
6. look at walks on higher cardinals using $\square$-sequences.

All of this is covered in Todorcevic's book [47], but we shall stick to one of the least technical topics which quickly leads to deep results: trees and linear orders. We will see plenty of the fundamental arguments about minimal walks and some highly non-trivial proofs about the so-called full code (denoted by ρ_0) and the lower trace of the walk.

Exercises and problems

Exercise 3.3.3. *Prove that any set of $(r-1)(s-1)+1$ numbers contains a monotonically increasing subsequence of length r or a monotonically decreasing subsequence of length s.*

Exercise 3.3.4. *Show that ω_1 with the order topology is not metrizable.*

Exercise 3.3.5. *Prove that the Sierpinski colouring witnesses $\omega_1 \not\to [\omega_1]^2_2$.*

Exercise 3.3.6. *Suppose that the colouring c witnesses $\omega_1 \not\to [\omega_1]^2_{\omega_1}$. Show that for any $\alpha < \omega_1$, there is an infinite $A \subset \omega_1$ so that $c \restriction [A]^2$ is constant α.*

Exercise 3.3.7. *Prove that any Aronszajn tree has a subtree with the property that each node has uncountably many extensions.*

Problem 3.3.8. *Prove that there is a subset T of $\{t \subset \mathbb{Q} : \max t \in \mathbb{Q}\}$ which forms an Aronszajn tree with the end-extension relation. Show that this T must be special.*

Problem 3.3.9. *Use an Aronszajn tree and Sierpinski's idea to witness $\omega_1 \not\to [\omega_1]^2_3$.*

Problem 3.3.10. *Prove that for any $c : \omega_1 \times \omega \to k$ with k finite, there are infinite $A \subset \omega_1, B \subset \omega$ so that $c \restriction A \times B$ is constant.*

Problem 3.3.11. *Show that if V is a $\mathfrak{c}^+$-dimensional vector space over $\mathbb{Q}$ and $c : V \to \omega$ then there is a monochromatic solution to $x + y = z$ with x, y, z pairwise distinct and non-zero.*

Problem 3.3.12. *For any colouring $c : \mathcal{P}(\omega) \to \omega$, there is a monochromatic, non-trivial solution to $X \cup Y = Z$.*

3.3.1 The full code of the walk

In the proof of the Expansion Lemma, we did the following: given $\delta \leq \beta$, we proved that for any large enough $\alpha < \delta$, $\delta \in \mathrm{Tr}(\alpha, \beta)$ and in fact, $\mathrm{Tr}(\delta, \beta) \subset \mathrm{Tr}(\alpha, \beta)$. We can actually quantify how large α needs to be: for $\delta < \beta$, we define

$$\lambda(\delta, \beta) = \max\{\max C_\xi \cap \delta : \xi \in \mathrm{Tr}(\delta, \beta)\}.$$

Now, the following fact holds.

Fact 3.3.13. *Suppose that $\delta < \beta$ and $\lambda(\delta, \beta) < \alpha < \delta$. Then $\mathrm{Tr}(\delta, \beta) \subset \mathrm{Tr}(\alpha, \beta)$ and in fact,*

$$\mathrm{Tr}(\alpha, \beta) = \mathrm{Tr}(\delta, \beta) \cup \mathrm{Tr}(\alpha, \delta).$$

Note that the reverse must hold too: if $\mathrm{Tr}(\delta, \beta) \subset \mathrm{Tr}(\alpha, \beta)$ then $\lambda(\delta, \beta) < \alpha$.

We shall continue with analysing the basic *characteristics* of minimal walks, the first one being the *full code of the walk* denoted by ρ_0. Now, $\rho_0(\alpha, \beta)$ is a finite sequence of natural numbers that simply records the sizes of the sets $C_{\beta_i} \cap \alpha$ along the walk $\beta = \beta_0 > \beta_1 > \cdots > \beta_n = \alpha$. That is, define

$$\rho_0 : [\omega_1]^2 \to \omega^{<\omega}$$

recursively by

$$\rho_0(\alpha, \beta) = \langle |C_\beta \cap \alpha| \rangle^\frown \rho_0(\alpha, \mathrm{step}(\alpha, \beta)).$$

Here the boundary condition is $\rho_0(\alpha, \alpha) = \emptyset$.

So, if $C_\beta(k)$ denotes the kth element of C_β in its increasing enumeration then

$$\mathrm{step}(\alpha, \beta) = C_\beta(\rho_0(\alpha, \beta)(0)).$$

Similarly, β_{i+1}, the $i + 1$st element of the walk is simply

$$\beta_{i+1} = C_{\beta_i}(\rho_0(\alpha, \beta)(i)).$$

So, given the starting node β, we can recover the minimal walk to α solely from this finite sequence of natural numbers (and the C-sequence).

We can also rephrase Fact 3.3.13 as follows.

Fact 3.3.14. *Suppose that $\alpha < \delta < \beta$. Then the following are equivalent:*

1. $\delta \in Tr(\alpha, \beta)$,
2. $\lambda(\delta, \beta) < \alpha$,
3. $\rho_0(\alpha, \beta) = \rho_0(\delta, \beta)^\frown \rho_0(\alpha, \delta)$

Now, a two place function like ρ_0 gives rise to a sequence of functions by its fibers: for $\beta < \omega_1$, we look at $\rho_{0\beta} : \beta \to \omega^{<\omega}$ which is defined by

$$\rho_{0\beta}(\alpha) = \rho_0(\alpha, \beta).$$

What can we say about these fibers?

Claim 3.3.15. *For any $\beta < \omega_1$, $\rho_{0\beta} : \beta \to \omega^{<\omega}$ is injective.*

Proof. Suppose that $\rho_0(\alpha, \beta) = \rho_0(\alpha', \beta)$ and that their common length is n. If $n = 0$ then $\alpha = \beta = \alpha'$ so we are done. Let $(\beta_i)_{i<n}$ and $(\beta'_i)_{i<n}$ denote the trace for the two walks from β to α and α'. Note that $\beta_0 = \beta = \beta'_0$. We shall prove that $\beta_i = \beta'_i$ for all $i < n$ by induction. However, this simply follows as $\beta_{i+1} = C_{\beta_i}(\rho_0(\alpha, \beta)(i)) = C_{\beta'_i}(\rho_0(\alpha', \beta)(i)) = \beta'_i$. □

The sequence of maps $(\rho_{0\beta})_{\beta<\omega_1}$ is not necessarily coherent but still, the tree

$$T(\rho_0) = \{\rho_{0\beta} \restriction \alpha : \alpha \leq \beta < \omega_1\}$$

will be have many interesting properties. In fact, $T(\rho_0)$ is a special Aronszajn tree (ordered by the subset relation). Special, in this setting, means that $T(\rho_0)$ is a countable union of antichains (i.e., pairwise incomparable nodes). The latter is equivalent to the existence of a strictly increasing map $a : T(\rho_0) \to \mathbb{Q}$ [44].

One of our main goals will be to see how to turn $T(\rho_0)$ into a *Countryman line* i.e., an uncountable linear order $C(\rho_0)$ with the property that the square of $C(\rho_0)$ is the union of countably many chains.

3.3.2 The simplified lower trace and $T(\rho_0)$

To understand the tree $T(\rho_0)$, we need to see how the fibers $\rho_{0\alpha}$ and $\rho_{0\beta}$ interact. That is, lets understand how two walks with the same destination, one from α to ζ and one from β to ζ, behave. These walks will start out with disjoint initial segments, then meet at some point ξ and then follow the same steps down to ζ; these points ξ will be critical. Define the *simplified lower trace* as follows: for $\alpha < \beta < \omega_1$, let

$$F(\alpha, \beta) = \{\xi \leq \alpha : \mathrm{Tr}(\xi, \alpha) \cap \mathrm{Tr}(\xi, \beta) = \{\xi\}\}.$$

In other words, we collect all those points ξ so that the walks from α to ξ and β to ξ only meet at ξ. This definition is due to W. R. Hudson [19] and lies between Todorcevic's original definition of the lower trace and full lower trace (which we will not define at this point).

So what is the first meeting point of two walks both going down to some ζ? Well, it is exactly the first element of the simplified lower trace above ζ.

Claim 3.3.16. *For any $\zeta < \alpha < \beta$, if $\xi = \min(F(\alpha, \beta) \setminus \zeta)$ then*

$$\rho_0(\zeta, \alpha) = \rho_0(\xi, \alpha) \cup \rho_0(\zeta, \xi)$$

and

$$\rho_0(\zeta, \beta) = \rho_0(\xi, \beta) \cup \rho_0(\zeta, \xi).$$

Proof. Suppose that $\delta = \max \mathrm{Tr}(\zeta, \alpha) \cap \mathrm{Tr}(\zeta, \beta)$. Then $\delta \in F(\alpha, \beta) \setminus \zeta$. Also, note that ζ and so ξ as well must be above $\lambda(\delta, \alpha)$ and $\lambda(\delta, \beta)$. If $\zeta \leq \xi < \delta$ then $\delta \in \mathrm{Tr}(\xi, \alpha) \cap \mathrm{Tr}(\xi, \beta)$, a contradiction to $\xi \in F(\alpha, \beta)$. $\square$

It is also easy to see now that the simplified lower trace is finite.

Claim 3.3.17. *For any $\alpha < \beta < \omega_1$, $F(\alpha, \beta)$ is finite.*

Proof. Suppose that $F(\alpha, \beta)$ has some accumulation point $\delta \leq \alpha$. Now, we can find $\xi \in F(\alpha, \beta) \cap \delta$ so that ξ is above $\lambda(\delta, \alpha)$ and $\lambda(\delta, \beta)$. However, this means that $\delta \in \mathrm{Tr}(\xi, \alpha) \cap \mathrm{Tr}(\xi, \beta) = \{\xi\}$, a contradiction. $\square$

Using the above, we shall prove that $T(\rho_0)$ has countable levels.[4]

Theorem 3.3.18. *$T(\rho_0)$ is an Aronszajn-tree.*

Proof. We need that $T(\rho_0)$ has countable levels and no uncountable chains. The latter we know already: since the fibers $\rho_{0\beta}$ are all 1-1, an uncountable chain through $T(\rho_0)$ would give a 1-1 map from ω_1 to the countable set $\omega^{<\omega}$.

Now, to show that all levels are countable, we need that for any fixed $\alpha < \omega_1$, the set

$$\{\rho_{0\beta} \restriction \alpha : \alpha \leq \beta < \omega_1\}$$

is countable. We will prove that the simplified lower trace $F(\alpha, \beta)$ and the behaviour of $\rho_{0\beta}$ on this finite set completely determines $\rho_{0\beta} \restriction \alpha$.

[4] Unlike the case of building trees from coherent sequences, this is not automatic now.

Claim 3.3.19. *Suppose that $\alpha \leq \beta < \beta'$, $F(\alpha, \beta) = F(\alpha, \beta')$ and if we denote the latter set by F, $\rho_{0\beta} \restriction F = \rho_{0\beta'} \restriction F$. Then*

$$\rho_{0\beta} \restriction \alpha = \rho_{0\beta'} \restriction \alpha.$$

Proof. Suppose that $\zeta < \alpha$ and let $\xi = \min(F \setminus \zeta)$. Then

$$\begin{aligned} \rho_0(\zeta, \beta) &= \rho_0(\xi, \beta)^\frown \rho_0(\zeta, \xi) & (3.3.1)\\ &= \rho_0(\xi, \beta')^\frown \rho_0(\zeta, \xi) & (3.3.2)\\ &= \rho_0(\zeta, \beta'). & (3.3.3) \end{aligned}$$

Indeed, we just applied Claim 3.3.16 twice. That is, $\rho_{0\beta} \restriction \alpha = \rho_{0\beta'} \restriction \alpha$. □

Since there are only countably many choices for the finite set F (from $[\alpha + 1]^{<\omega}$) and the restriction $\rho_{0\beta} \restriction F$, the αth level in $T(\rho_0)$ must be countable. In turn, we proved that $T(\rho_0)$ is an Aronszajn tree.

⊠

The simplified lower trace also satisfies the following triangle inequality which is useful in many applications.

Claim 3.3.20. *If $\zeta < \alpha < \beta$ then $F(\zeta, \alpha) \subset F(\zeta, \beta) \cup F(\alpha, \beta)$.*

We omit the proof for now.

Now, we would actually like to prove that $T(\rho_0)$ is special. First, the set $\omega^{<\omega}$ can be regarded as a copy of the rationals. More precisely, define the *right lexicographic ordering* $<_\ell$ on $\omega^{<\omega}$ as follows: $s <_\ell t$ if $s \supsetneq t$ or $s(j) < t(j)$ for the minimal j so that $s(j) \neq t(j)$.

Now, $\emptyset$ is the $<_\ell$-largest element of $\omega^{<\omega}$ and for any $s \in \omega^{<\omega}$, the sequence of successors $(s^\frown < n >)_{n\in\omega}$ will be a $<_\ell$-increasing sequence that converges to s. In turn,

$$(\omega^{<\omega}, <_\ell) \cong (0, 1] \cap \mathbb{Q}$$

with the usual order. Following [47], and for notational simplicity, we shall denote this set by $\mathbb{Q}_r$.

Now, we prove that all the fibers of ρ_0 define monotone increasing maps into $\mathbb{Q}_r$.

Claim 3.3.21. *For any $\beta < \omega_1$, $\rho_{0\beta} : \beta \to \omega^{<\omega}$ is monotone increasing.*

Proof. Let $\alpha < \alpha' < \beta$. If $\rho_0(\alpha, \beta) \supseteq \rho_0(\alpha', \beta)$ then $\rho_0(\alpha, \beta) <_\ell \rho_0(\alpha', \beta)$ as desired.

Let $(\beta_i)_{i<n}$ and $(\beta'_i)_{i<n'}$ denote the trace for the two walks from β to α and α'. If j is the first place where $\rho_0(\alpha, \beta)$ and $\rho_0(\alpha', \beta)$ differ then $\beta_j = \beta'_j$ and but $\beta_{j+1} < \alpha'$. In turn,

$$\rho_0(\alpha, \beta)(j) = |C_{\beta_j} \cap \alpha| < |C_{\beta_j} \cap \alpha'| = |C_{\beta'_j} \cap \alpha'| = \rho_0(\alpha', \beta)(j).$$

So $\rho_0(\alpha, \beta) <_\ell \rho_0(\alpha', \beta)$. □

Let us also show that $T(\rho_0)$ does not branch at limit levels, which follows from the next claim.

Claim 3.3.22. *For any $\beta < \beta'$, $\{\xi < \beta : \rho_0(\xi, \beta) = \rho_0(\xi, \beta')\}$ is closed in β.*

Proof. Suppose that δ is an accumulation point of $E = \{\xi < \beta : \rho_0(\xi, \beta) = \rho_0(\xi, \beta')\}$. So, we can pick $\xi \in E \cap \delta$ which is larger than both $\lambda(\delta, \beta)$ and $\lambda(\delta, \beta')$. Now $\rho_0(\xi, \beta) = \rho_0(\delta, \beta)^\frown \rho_0(\xi, \delta)$ and $\rho_0(\xi, \beta') = \rho_0(\delta, \beta')^\frown \rho_0(\xi, \delta)$. Since $\rho_0(\xi, \beta) = \rho_0(\xi, \beta')$ we must have $\rho_0(\delta, \beta) = \rho_0(\delta, \beta')$ as well. So $\delta \in E$, as desired.

$\square$

It clearly follows now that if $\rho_{0\beta} \restriction \alpha = \rho_{0\beta'} \restriction \alpha$ then $\rho_{0\beta}(\alpha) = \rho_{0\beta'}(\alpha)$ and so $\rho_{0\beta} \restriction \alpha + 1 = \rho_{0\beta'} \restriction \alpha + 1$. Also, note that for any limit α,

$$\rho_{0\beta}(\alpha) = \sup_{<_\ell} \rho_{0\beta} \restriction \alpha.$$

Indeed, if ξ is the $n+1$st element of C_α then $\rho_0(\xi, \beta) = \rho_0(\alpha, \beta)^\frown \langle n \rangle$ and the latter sequence increasingly converges to $\rho_0(\alpha, \beta)$.

Corollary 3.3.23. *$T(\rho_0)$ is special.*

Proof. We can define a monotone map f from $T(\rho_0)$ to $\mathbb{Q}_r$ as follows: if $t \in T(\rho_0)$ of level α and t is terminal then let $f(t) = 1$. If t is not terminal then pick any $\beta > \alpha$ so that $t = \rho_{0\beta} \restriction \alpha$ and let $f(t) = \rho_\beta(\alpha)$. Claim 3.3.22 shows that $f(t)$ is well defined (it does not matter which β we choose, we get the same value). Now, suppose that $t < t'$ are non terminal nodes and let $f(t) = \rho_{0\beta}(\alpha)$ and $f(t) = \rho_{0\beta'}(\alpha')$. We can assume that $\beta = \beta'$ and so $f(t) < f(t')$ by Claim 3.3.21.

$\boxtimes$

3.4 Countryman lines

Look at the tree $T(\rho_0)$ which consists of functions from a countable ordinal to $\omega^{<\omega}$ i.e., well ordered sequences from $\mathbb{Q}_r$. The latter set is linearly ordered by the right lexicographical order $<_\ell$. So, we can repeat the lexicographic construction and turn the tree $T(\rho_0)$ itself into a linear order: let $s <_\ell t$ if $s \supset t$ or $s(\delta) <_\ell t(\delta)$ in ω^ω for $\delta = \min\{\xi : s(\xi) \neq t(\xi)\}$. We shall denote the set $\{\rho_{0\beta} : \beta < \omega_1\}$ with this lexicographic linear order by $C(\rho_0)$.

Finally, we arrived at the main result of this chapter. Note that for any linear order L, L^2 can be regarded as a partial order: $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$. To see an example, consider $L = \mathbb{R}$ with the usual order. Looking at the poset L^2 with this coordinate-wise order, we see that $\{(x, x) : x \in \mathbb{R}\}$ is an uncountable chain and $\{(x, -x) : x \in \mathbb{R}\}$ is an uncountable antichain; indeed if $x < x'$ then $-x' < -x$ so no two elements of the latter set are comparable.

Theorem 3.4.1. *$C(\rho_0)$ is a Countryman-line i.e., the square of $C(\rho_0)$ is the union of countably many chains.*

It is quite surprising such objects can exist: the only trivial chains we see in the square are the diagonal and the horizontal/vertical lines which do not really help a countable cover. Yet somehow, the square can be covered by countably many chains. Let us also point out two further facts (without a proof at this point):

- Any Countryman line is necessarily an Aronszajn line i.e., has no uncountable real suborder or copy of $\pm\omega_1$.

- If C is Countryman then C and $-C$ have no common uncountable suborder.

Note that if C is a Countryman line then so does $-C$.

Proof of Theorem 3.4.1. First of all, $C = C(\rho_0)$ can be regarded as a linear order on ω_1 by the map $\alpha \mapsto \rho_{0\alpha}$. To show that C^2 is the union of countably many chains, we need to assign an invariant $\sigma(\alpha, \beta)$ to all pairs $\alpha < \beta \in \omega_1$ so that

1. all the invariants come from a fixed countable set, and
2. pairs with the same invariant are comparable in C^2: if $\sigma(\alpha, \beta) = \sigma(\alpha', \beta')$ then

$$\rho_{0\alpha} <_\ell \rho_{0\alpha'} \Leftrightarrow \rho_{0\beta} <_\ell \rho_{0\beta'}.$$

The above will achieve that $\{(\alpha, \beta) : \alpha < \beta \in \omega_1\}$ is the union of countably many chains (the part of C^2 above the diagonal). The latter set is order isomorphic to $\{(\beta, \alpha) : \alpha < \beta \in \omega_1\}$ (the lower half of C^2) so its the union of countably many chains as well. All that is left of C^2 is the diagonal $\{(\alpha, \alpha) : \alpha \in \omega_1\}$ which is clearly a chain.

Following this plan (and keeping in mind the proof of $T(\rho_0)$ being Aronszajn), σ will record the behaviour of $\rho_{0\alpha}$ and $\rho_{0\beta}$ on the finite set $F(\alpha, \beta)$ i.e., $\sigma = \sigma(\alpha, \beta)$ is defined to be

(i) a finite sequence of pairs form $\omega^{<\omega}$,

(ii) the length of σ is $n = |F(\alpha, \beta)|$,

(iii) if $i < n$ and ξ is the ith element of $F(\alpha, \beta)$ then

$$\sigma(i) = (\rho_0(\xi, \alpha), \rho_0(\xi, \beta)).$$

Now, assume that $\sigma(\alpha, \beta) = \sigma(\alpha', \beta')$ with common value σ. Let n denote the common size of $F = F(\alpha, \beta)$ and $F' = F(\alpha', \beta')$ and let $\{\xi_i : i < n\}$ and $\{\xi'_i : i < n\}$ be their increasing enumeration. Assume that j is minimal so that $\xi_j \neq \xi'_j$ and we shall assume that $\xi_j < \xi'_j$ (we did not assume anything about the ordering between α, β and α', β' so we do not loose any generality by assuming $\xi_j < \xi'_j$ here). Recall that

$$\rho_{0\alpha}(\xi_i) = \rho_{0\alpha'}(\xi'_i) \text{ and } \rho_{0\beta}(\xi_i) = \rho_{0\beta'}(\xi'_i)$$

for all $i < n$ (by our assumption on the invariant σ).

The bulk of proving the theorem is in the following claim.

Claim 3.4.2. *$\Delta(\alpha, \alpha') = \Delta(\beta, \beta')$ and this common value falls between $\xi_{j-1} = \xi'_{j-1}$ and ξ_j.*

Proof. First, lets walk from α' to ξ_j. Using that $\xi'_j = \min F' \setminus \xi_j$,

$$\rho_0(\xi_j, \alpha') = \rho_0(\xi'_j, \alpha')^\frown \rho_0(\xi_j, \xi'_j) \tag{3.4.1}$$

$$= \sigma(j)(0)^\frown \rho_0(\xi_j, \xi'_j) \tag{3.4.2}$$

$$= \rho_0(\xi_j, \alpha)^\frown \rho_0(\xi_j, \xi'_j) \neq \rho_0(\xi_j, \alpha). \tag{3.4.3}$$

So $\Delta(\alpha, \alpha') \leq \xi_j$.

Similarly, now walking from β' to ξ_j we see that

$$\rho_0(\xi_j, \beta') = \rho_0(\xi_j', \beta')^\frown \rho_0(\xi_j, \xi_j') \tag{3.4.4}$$
$$= \sigma(j)(1)^\frown \rho_0(\xi_j, \xi_j') \tag{3.4.5}$$
$$= \rho_0(\xi_j, \beta)^\frown \rho_0(\xi_j, \xi_j') \neq \rho_0(\xi_j, \beta). \tag{3.4.6}$$

So $\Delta(\beta, \beta') \leq \xi_j$.

On the other hand, if $\zeta \leq \xi_{j-1} = \xi_{j-1}'$ the there is some $k < j$ so that

$$\xi_{k-1} = \xi_{k-1}' < \zeta \leq \xi_k = \xi_k'.$$

Note that

$$\min F \setminus \zeta = \xi_k = \xi_k' = \min F' \setminus \zeta$$

so, walking from α to ζ, the full code is

$$\rho_0(\zeta, \alpha) = \rho_0(\xi_k, \alpha)^\frown \rho_0(\zeta, \xi_k) \tag{3.4.7}$$
$$= \sigma(k)(0)^\frown \rho_0(\zeta, \xi_k) \tag{3.4.8}$$
$$= \rho_0(\xi_k', \alpha')^\frown \rho_0(\zeta, \xi_k') \tag{3.4.9}$$
$$= \rho_0(\zeta, \alpha'). \tag{3.4.10}$$

In turn, $\Delta(\alpha, \alpha') \geq \xi_{j-1}$.

Similarly for β,

$$\rho_0(\zeta, \beta) = \rho_0(\xi_k, \beta)^\frown \rho_0(\zeta, \xi_k) \tag{3.4.11}$$
$$= \sigma(k)(1)^\frown \rho_0(\zeta, \xi_k) \tag{3.4.12}$$
$$= \rho_0(\xi_k', \beta')^\frown \rho_0(\zeta, \xi_k') \tag{3.4.13}$$
$$= \rho_0(\zeta, \beta'). \tag{3.4.14}$$

In turn, $\Delta(\beta, \beta') \geq \xi_{j-1}$.

So let $\delta_1 = \Delta(\alpha, \alpha')$ and let $\delta_2 = \Delta(\beta, \beta')$. We proved already that $\delta_1, \delta_2 \in (\xi_{j-1}, \xi_j]$. Note that the walks from α to δ_1 and α' to δ_1 give different full codes i.e.,

$$\rho_0(\xi_j, \alpha)^\frown \rho_0(\delta_1, \xi_j) = \rho_0(\delta_1, \alpha) \tag{3.4.15}$$
$$\neq \rho_0(\delta_1, \alpha') = \rho_0(\xi_j', \alpha')^\frown \rho_0(\delta_1, \xi_j') \tag{3.4.16}$$

Both sequences start out with $\sigma(j)(0)$, so it must be that $\rho_0(\delta_1, \xi_j) \neq \rho_0(\delta_1, \xi_j')$. This in turn, implies that if we walk from β and β' to δ_1 then

$$\rho_0(\delta_1, \beta) = \rho_0(\xi_j, \beta)^\frown \rho_0(\delta_1, \xi_j) \tag{3.4.17}$$
$$\neq \rho_0(\xi_j', \beta')^\frown \rho_0(\delta_1, \xi_j') = \rho_0(\delta_1, \beta'). \tag{3.4.18}$$

This shows that $\delta_2 \leq \delta_1$ since $\rho_{0\beta}$ and $\rho_{0\beta'}$ already differ on δ_1. A symmetrical argument proves that $\delta_1 \leq \delta_2$ (first walk from β, β' to δ_2 to see that $\rho_0(\delta_2, \xi_j) \neq \rho_0(\delta_2, \xi_j')$ which in turn, will imply that $\rho_{0\alpha}(\delta_2) \neq \rho_{0\alpha'}(\delta_2)$). So, we showed that $\delta_1 = \delta_2$, as desired.

□

Let δ denote this common value $\Delta(\alpha, \alpha') = \Delta(\beta, \beta')$. Looking at the walks to δ and applying the Main Claim 3.3.16, we see that

$$\rho_0(\delta, \alpha) = \rho_0(\xi_j, \alpha)^\frown \rho_0(\delta, \xi_j) \tag{3.4.19}$$
$$= \sigma(j)(0)^\frown \rho_0(\delta, \xi_j) \tag{3.4.20}$$

and

$$\rho_0(\delta, \alpha') = \rho_0(\xi'_j, \alpha')^\frown \rho_0(\delta, \xi'_j) \tag{3.4.21}$$
$$= \sigma(j)(0)^\frown \rho_0(\delta, \xi'_j). \tag{3.4.22}$$

So, if we know that the relation $<_\ell$ between $\rho_{0\alpha}$ and $\rho_{0\alpha'}$ is decided on the final segments after $\sigma(j)(0)$ i.e.,

$$\rho_0(\delta, \alpha) <_\ell \rho_0(\delta, \alpha') \Leftrightarrow \rho_0(\delta, \xi_j) <_\ell \rho_0(\delta, \xi'_j). \tag{3.4.23}$$

Similarly, looking at β, β' instead of α, α',

$$\rho_0(\delta, \beta) = \rho_0(\xi_j, \beta)^\frown \rho_0(\delta, \xi_j) \tag{3.4.24}$$
$$= \sigma(j)(1)^\frown \rho_0(\delta, \xi_j) \tag{3.4.25}$$

and

$$\rho_0(\delta, \beta') = \rho_0(\xi'_j, \beta')^\frown \rho_0(\delta, \xi'_j) \tag{3.4.26}$$
$$= \sigma(j)(1)^\frown \rho_0(\delta, \xi'_j). \tag{3.4.27}$$

Since $\rho_0(\delta, \beta)$ and $\rho_0(\delta, \beta')$ have the common initial segment $\sigma(j)(1)$, their lexicographic relation is decided by the relation between $\rho_0(\delta, \xi_j)$ and $\rho_0(\delta, \xi'_j)$. That is,

$$\rho_0(\delta, \beta) <_\ell \rho_0(\delta, \beta') \Leftrightarrow \rho_0(\delta, \xi_j) <_\ell \rho_0(\delta, \xi'_j). \tag{3.4.28}$$

Combining the above, we see that

$$\rho_0(\delta, \alpha) <_\ell \rho_0(\delta, \alpha') \Leftrightarrow \rho_0(\delta, \beta) <_\ell \rho_0(\delta, \beta') \tag{3.4.29}$$

as desired.

$\boxtimes$

Finally, instead of $(\rho_{0\beta})_{\beta \in \omega_1}$ (which was 1-1 but not necessarily coherent), one can use *any coherent sequence of finite-to-one maps* to get Countryman lines [47].

Theorem 3.4.3. *Suppose that $a_\alpha : \alpha \to \omega$ is finite-to-one and $a_\alpha =^* a_\beta \restriction \alpha$ for all $\alpha < \beta < \omega_1$. Then $(T(a), <_\ell)$ is a Countryman line.*

As before, the proof depends on defining an adequate variant of the simplified lower trace now based on the coherent sequence. Let us emphasize again that ρ_0 is not necessarily coherent so the above result on $C(\rho_0)$ is independent of this latter theorem.

Exercises and problems

Exercise 3.4.4. *Suppose that P is a partial order and any antichain in P has finite size at most k. Show that P is the union of at most k chains given that all its finite subsets satisfy this property.*

Let L be a *Countryman line* i.e., a linear order which is uncountable but L^2 is the union of countably many chains (partially ordered by the coordinate-wise order). Without knowing whether such things could exist, let's prove a few things.

Exercise 3.4.5. *Show that L has no uncountable well-ordered or reverse well-ordered subset.*

Problem 3.4.6. *Show that L has no uncountable suborder isomorphic to a set of reals.*

Problem 3.4.7. *Show that L and its reverse $-L$ has no common uncountable suborder.*

The *maximal weight* is an important characteristic of minimal walks: define

$$\rho_1 : [\omega_1]^2 \to \omega$$

by

$$\rho_1(\alpha, \beta) = \max\{|C_\xi \cap \alpha| : \xi \in Tr(\alpha, \beta)\}.$$

In other words, $\rho_1(\alpha, \beta) = \max \rho_0(\alpha, \beta)$.

Problem 3.4.8 (Finite-to-one property). *Show that for any $\beta < \omega_1$ and $n < \omega$, the set*

$$\{\alpha < \beta : \rho_1(\alpha, \beta) \leq n\}$$

is finite.

Problem 3.4.9 (Coherence of max. weight). *Show that for any $\alpha < \beta < \omega_1$, the set*

$$\{\xi < \alpha : \rho_1(\xi, \beta) \neq \rho_1(\xi, \alpha)\}$$

is finite.

The *number of steps* is another characteristic of minimal walks: define

$$\rho_2 : [\omega_1]^2 \to \omega$$

by

$$\rho_2(\alpha, \beta) = |Tr(\alpha, \beta)| - 1.$$

Problem 3.4.10 (Semi-coherence of number of steps). *Show that for any $\alpha < \beta < \omega_1$,*

$$\sup_{\xi < \alpha} |\rho_2(\xi, \beta) - \rho_2(\xi, \alpha)| < \infty.$$

3.4.1 The basis problem for uncountable linear orders

Recall that any infinite linear order contains a copy of ω or its reverse. In other words, the class of infinite linear orders has a 2-element basis. In this section, we focus on the class of uncountable linear orders and analyse if there is a small basis i.e., is there a small set $\mathcal{B}$ of uncountable linear orders so that any uncountable linear order L will embed some element of $\mathcal{B}$. There is always a basis of size $2^{\aleph_1}$ but can we do better?

First, let's find a lower bound for the size of a basis. We say that two uncountable linear orders L, K are *orthogonal* if they have no common uncountable linear orders. For example, ω_1 and $-\omega_1$ are orthogonal or C and $-C$ for any Countryman line C. Recall that *Aronszajn lines* by definition are linear orders which are orthogonal both to $\pm\omega_1$ and $\mathbb{R}$.

So, there are three groups of pairwise orthogonal uncountable linear orders:

1. the two-element class of ω_1 and $-\omega_1$,
2. subsets of $\mathbb{R}$, and
3. Aronszajn lines.

In principle, $\mathbb{R}$ can have a 1-element basis but we need at least 2 orders from the Aronszajn lines (because a Countryman line and its reverse are orthogonal). So any basis must have at least 5 elements.

We shall see that under certain assumptions (like the Proper Forcing Axiom or PFA, in short) this is possible and there is indeed a 5-element basis to all uncountable linear orders. On the under hand, under CH, there is no basis of size $< 2^{\aleph_1}$ even for the real suborders.

Models of PFA

The first clue that a small basis is conceivable was Baumgartner's following theorem from 1973.

Theorem 3.4.11. *[5] Under PFA, any two $\aleph_1$-dense sets of reals are isomorphic.*[5]

This implies that any set of reals of size $\aleph_1$ forms a 1-element basis for uncountable separable orders. The above theorem can be achieved by ccc forcing over a model of CH as well, so no large cardinals are required for this result. However, the proof of this theorem is far from standard and makes an elaborate and ingenious use of elementary submodels. The naive forcing that approximates an order isomorphism by finite conditions easily fails to be ccc. So one needs to space out these approximations (by elementary submodels) to allow the amalgamation arguments to work.

It was later proved that $MA_{\aleph_1}$ does not suffice to deduce the above result [2].

After the discovery of Countryman lines, it was a long standing open problem of Shelah if the Aronszajn orders can have a 2-element basis. First, Todorcevic proved that already under $MA_{\aleph_1}$, the Countryman line $C(\rho_0)$ is very special.

Theorem 3.4.12. *Under $MA_{\aleph_1}$, any Countryman line embeds a copy of $C(\rho_0)$ or its reverse.*

[5] $\aleph_1$-dense means that any non-empty interval has size $\aleph_1$.

So Countryman lines, which are certain special Aronszajn lines, have a 2-element basis (under $MA_{\aleph_1}$). How about other Aronszajn lines? Well, it turns out that consistently, any Aronszajn order must contain a Countryman suborder and in fact a copy of $C(\rho_0)$ or its reverse. Indeed, Shelah's problem was resolved by Justin Moore in 2006 and in turn, he proved the consistency of the 5-element basis.

Theorem 3.4.13. *[34] Assume PFA. If X is any set of reals of size $\aleph_1$ and C is a Countryman line then $\{\pm\omega_1, \pm C, X\}$ forms a 5-element basis for uncountable linear orders.*

It is not known if the large cardinal assumption hidden in PFA can be removed from the consistency of the 5-element basis result. It is known that a Mahlo cardinal suffices though.

Moore's proof hinges on an interesting combinatorial lemma about Aronszajn trees.

Theorem 3.4.14. *Suppose PFA. Then there is an Aronszajn tree T so that for any $K \subset T$, there is an uncountable antichain $A \subset T$ so that either $\wedge(A) \subset K$ or $\wedge(A) \cap K = \emptyset$.*

Here, the tree T consist of functions from countable ordinals to ω and for $s, t \in T$, $s \wedge t$ is their longest common initial part. The notation $\wedge(A)$ stands for $\{s \wedge t : s \neq t \in A\}$. Very roughly, the theorem says that in this particular Aronszajn tree, we have a weak measure over arbitrary subsets.

Models of CH

Under CH, one can construct many orthogonal linear orders:

1. [38, Sierpinski, 1932] there are $2^{\aleph_1}$ many pairwise orthogonal suborders of the reals;
2. [1, Abraham-Shelah, 1985] there are $2^{\aleph_1}$ many pairwise orthogonal Aronszajn lines.

In fact, the former result essentially follows from the first theorem we proved in the class about rigid suborders while the Abraham-Shelah result is somewhat more involved. In any case, there is no chance of finding a small basis by any reasonable measure under CH.

Minimal linear orders

If you have a basis for the uncountable linear orders consisting of pairwise orthogonal elements then they must all be *minimal* i.e., they embed into all their uncountable suborders.

For example, ω and $-\omega$ are the only countably infinite minimal linear orders. Similarly, regardless of the set theoretic axioms, $\pm\omega_1$ are always minimal. Under PFA, by Baumgartner's result, any $\aleph_1$-dense set of reals is minimal. Similarly, Countryman lines are also minimal under PFA.

Under CH, however, no uncountable set of reals is minimal. Moreover, J. Moore proved that there are models of CH where the only uncountable minimal orders are $\pm\omega_1$ [33]. His argument splits into two parts: characterizing minimal Aronszajn orders using ladder system uniformization on trees and then iterating a non-trivial set of forcings to uniformize certain colourings while preserving CH. Such arguments appeared in a very different setting in Shelah's work on the Whitehead problem.

The basis problem for uncountable topological spaces

It is an easy exercise to show that any infinite, Hausdorff topological space embeds $D(\omega)$, the countably infinite discrete space. So we have a 1-element basis for all infinite topological spaces. How about uncountable spaces? We need to restrict our attention to hope for a reasonable solution but not much as the following beautiful problem of G. Gruenhage is still open.

Problem 3.4.15. *[16] Is it consistent that any uncountable, first-countable and regular space contains either a copy of $D(\omega_1)$, a fixed subset of $\mathbb{R}$ or a fixed subset of the Sorgenfrey line.*

That is, can we have a 3-element basis for uncountable, first-countable regular spaces? Recall that the Sorgenfrey line is the topology on $\mathbb{R}$ generated by the half open intervals $(a, b]$ for $a < b \in \mathbb{R}$. Again, PFA (or Martin's Maximum, a strengthening of PFA) is a reasonable candidate to settle this problem. The assumption of first-countability cannot be dropped as Moore's L-space construction gives $2^{\aleph_1}$-many pairwise orthogonal (0-dimensional, Hausdorff) topological spaces [32].

Exercises and problems

Exercise 3.4.16. *Suppose that $X \subset \mathbb{R}$ has size $\aleph_1$. Show that there is a countable $A \subset X$ so that $X \setminus A$ is $\aleph_1$-dense.*

Exercise 3.4.17. *Suppose that L is a linear order so that L and $-L$ has no common infinite suborder. Prove that either L or $-L$ is well ordered.*

Exercise 3.4.18. *Prove that a Countryman line L cannot be ccc i.e., there must be an uncountable collection of pairwise disjoint non-empty intervals in L.*

Problem 3.4.19. *Show that for any cardinal κ, there is a linear order of size κ which has more than κ initial segments.*

Problem 3.4.20. *Use the Continuum Hypothesis e.g., $2^{\aleph_0} = \aleph_1$ to construct a universal linear order L of size $\aleph_1$ inside $(\omega^\omega, <^*)$. That is, L has size $\aleph_1$ and embeds any linear order of size $\aleph_1$.*

We use the usual notation from the lectures regarding minimal walks along a $\underline{C}$-sequence on ω_1.

Exercise 3.4.21. *Prove that for any $\alpha < \beta$, $\max(F(\alpha, \beta) \cap \alpha) = \lambda(\alpha, \beta)$.*

Problem 3.4.22. *Prove that for any $\zeta < \alpha < \beta$, $F(\zeta, \alpha) \subset F(\zeta, \beta) \cup F(\alpha, \beta)$.*

Finally, some questions about partial orders.

Problem 3.4.23. *Let P be a partial order and let σP denote the set of well-ordered subsets of P ordered by end-extension. Show that there is no strictly increasing map from σP to P.*

Problem 3.4.24. *Show that there is a poset of size $\mathfrak{c}$ in which every chain and anti-chain is countable*

Problem 3.4.25. *Assume that $S, T \subset \omega_1$ so that $S \setminus T$ is stationary. Let σS and σT denote the set of closed subsets of S and T, respectively. Prove that there is no embedding of σS into σT.*

Problem 3.4.26. *Using the previous problem, show that there are $2^{\aleph_1}$-many pairwise non-embedable partial orders of size $\mathfrak{c}$.*

Chapter 4

Construction schemes

In this chapter, we review various ideas to construct a large object by one small piece or local approximation at a time. First, we look at a recent technique using finite approximations; as the simplest demonstration, we use this method to construct a coherent sequence of 1-1 maps. Then we shall see two other techniques (Davies trees and Kurepa families) that use countable approximating sets. Finally, we look at a new version of the Davies tree technique which uses countably closed elementary models of size continuum. We will re-prove some of the results about almost disjoint families and give a simple proof of a deep topological Ramsey result of. W. Weiss [49]: consistently, any Hausdorff space X has a colouring f with $\mathfrak{c}$ colours so that $f[C] = \mathfrak{c}$ on all copies C of the Cantor set in X.

4.1 Finite approximations

The first tool we review is a new technique due to S. Todorcevic [48] and we follow the exposition by F. Lopez [29]. This is a general framework for building combinatorial structures on ω_1 by gluing together larger and larger finite pieces each resembling the final object. The arguments will be similar to classical ccc forcing arguments that use finite conditions and amalgamations of Δ-systems.

4.1.1 Todorcevic's construction scheme

A construction scheme on ω_1 is a family $\mathcal{F} \subset [\omega_1]^{<\omega}$ with the following properties:

1. $\mathcal{F}$ is cofinal i.e., for any finite $E \subset \omega_1$ there is $F \in \mathcal{F}$ so that $E \subset F$;
2. $\mathcal{F} = \bigcup_{k<\omega} \mathcal{F}_k$ so that $|F| = m_k$ for all $F \in \mathcal{F}_k$ and $m_0 = 1$;
3. if $E, F \in \mathcal{F}_k$ then $E \cap F \sqsubseteq E, F$;
4. for any $F \in \mathcal{F}_k$ with $k \geq 1$ there is a unique decomposition $F = \bigcup_{i<n_k} F_i$ so that
   (a) $F_i \in \mathcal{F}_{k-1}$,
   (b) there is an $R(F)$ of size r_k so that $(F_i)_{i<n_k}$ forms an increasing Δ-system with root $R(F)$:
$$R(F) < F_0 \setminus R(F) < F_1 \setminus R(F) < \cdots < F_{n_{k-1}} \setminus R(F).$$

The latter is called the canonical decomposition of F. Note that we must have

$$m_k = r_k + n_k(m_{k-1} - r_k)$$

for all $k < \omega$. We call the sequences $\bar{m}, \bar{n}, \bar{r}$ the type of the construction scheme.

Construction schemes can be used to build various gaps, trees and topological and Banach spaces. Moreover, they exist in ZFC.

Theorem 4.1.1. *[48] For any type $\bar{m}, \bar{n}, \bar{r}$ that satisfies $m_k = r_k + n_k(m_{k-1} - r_k)$, there is a construction scheme $\mathcal{F}$ of that type.*

We will not prove this theorem but continue with some basic properties of construction schemes and a simple application.

First, let's try to understand better how these finite sets interact.

Lemma 4.1.2. *If $F \in \mathcal{F}_k, E \in \mathcal{F}_l$ and $l \leq k$ then $E \cap F \sqsubseteq E$.*

Proof. This is a double induction first on k and then on l. For $l = k$, the claim always holds by the definition of construction schemes; similarly, for $l = 0$ we have nothing to prove. Now, assume $F = \bigcup F_i \in \mathcal{F}_k$ and $E \in \mathcal{F}_l$ with $l < k$. We know by induction that $E \cap F_i \sqsubseteq E$ for all $i < n_k$. So there could be at most one $i < n_k$ so that $E \cap F_i \setminus R(F) \neq \emptyset$. In turn, $E \cap F_i \sqsubseteq E$ as desired.

□

Corollary 4.1.3. *Suppose that $F \in \mathcal{F}_k, E \in \mathcal{F}_l$ and F has canonical decomposition $\cup F_i$.*

1. *If $E \subset F$ and $l < k$ then $E \subset F_i$ for some $i < n_k$; if $l = k - 1$ then $E = F_i$.*
2. *If $k = l$ and $\varphi_{E,F} : E \to F$ is the unique order preserving map from E to F then $\varphi_{E,F}(\mathcal{F} \upharpoonright E) = \mathcal{F} \upharpoonright F$.*

Here, $\mathcal{F} \upharpoonright E = \{K \in \mathcal{F} : K \subseteq E\}$. So the finite structures $(E, \mathcal{F} \upharpoonright E, <)$ and $(F, \mathcal{F} \upharpoonright F, <)$ are isomorphic.

4.1.2 Coherent maps from construction schemes

Our goal is to demonstrate this method very briefly without getting too technical. So we construct something that we are familiar with already: a coherent sequence $\rho_\alpha : \alpha \to \omega$ of 1-1 maps.

The idea is that each infinite map $\rho_\alpha : \alpha \to \omega$ will be approximated by finite 1-1 maps

$$\rho^F_\alpha : F \cap \alpha \to N_k$$

where $\alpha \in F \in \mathcal{F}_k$. In the end, we would like to take

$$\rho_\alpha = \bigcup_{\alpha \in F \in \mathcal{F}} \rho^F_\alpha.$$

In order for this to work, we assume the following:

1. If $F \in \mathcal{F}_k, E \in \mathcal{F}_l$ and $l < k$ then

(a) $\rho^F_\alpha \restriction E = \rho^E_\alpha$,

(b) if $\beta \in F$ and $\xi < \alpha < \beta$ so that $\rho^F_\alpha(\xi) \neq \rho^E_\alpha(\xi)$ then $\xi \in E$.

2. If $E, F \in \mathcal{F}_k$ and $\beta = \varphi_{EF}(\alpha)$ then

$$\rho^E_\alpha = \rho^F_\beta \circ \varphi_{EF}.$$

Let's prove first that such a collection of finite maps gives the desired (ρ_α) sequence. First, why is $\rho_\alpha = \bigcup_{\alpha \in F \in \mathcal{F}} \rho^F_\alpha$ well defined? Assume that $F \in \mathcal{F}_k, E \in \mathcal{F}_l$ with $l \leq k$ and $\xi < \alpha \in E \cap F$. Find $E' \supset E$ so that $E' \in \mathcal{F}_k$ as well. Now, $\varphi_{E'F}$ must fix both α and ξ so $\rho^E_\alpha(\xi) = \rho^{E'}_\alpha(\xi) = \rho^F_\alpha(\xi)$.

Each ρ_α must be 1-1 since it is the union of 1-1 maps. Moreover, $\mathcal{F}$ is cofinal so dom $\rho_\alpha = \alpha$. Finally, suppose that there is an infinite $\xi_0 < \xi_1 < \dots$ below $\alpha < \beta$ so that $\rho_\alpha(\xi_n) \neq \rho_\beta(\xi_n)$ for all $n < \omega$. Find some $E \in \mathcal{F}$ so that $\alpha, \beta \in E$ and some ξ_N not in E. There is a larger $F \supseteq E$ so that $\xi_N \in F$ already. But now $\rho^F_\alpha(\xi_N) \neq \rho^F_\beta(\xi_N)$ but $\xi_N \notin E$, a contradiction to our construction.

So we are left to prove that the finite approximations $(\rho^F_\alpha)_{\alpha \in F \in \mathcal{F}}$ exist. We'll do this by induction on $\mathcal{F}_k$. Since each $F \in \mathcal{F}_0$ is a singleton, the maps ρ^F_α must be empty simply. Now assume that $(\rho^F_\alpha)_{\alpha \in F \in \mathcal{F}_{k-1}}$ are all constructed and take some $F \in \mathcal{F}_k$ with canonical decomposition $\bigcup F_i$.

If $\alpha \in F_j$ then on one hand, we need that $\rho^F_\alpha \restriction F_j = \rho^{F_j}_\alpha$. How do we extend this to $F \cap \alpha$? We need to define $\rho^F_\alpha(\xi)$ for $\xi \in F_i \setminus R(F)$ for $i < j$. We simply set

$$\rho^F_\alpha(\xi) = N_{k-1} + |F \cap \xi|.$$

So, $N_k = N_{k-1} + |F|$. First, note that ρ^F_α will be clearly 1-1. Moreover, if $\alpha \in F_i, \beta \in F_j$ for $i < j$ then $\rho^F_\alpha(\xi) \neq \rho^F_\beta(\xi)$ implies that $\xi \in F_i$. It follows that all our properties are preserved.

4.1.3 Capturing construction schemes

Using some additional assumptions, one can have construction schemes with some additional properties that allow much control over arbitrary uncountable subsets of the limit structures.

We say that a construction scheme $\mathcal{F}$ is *n-capturing* if for any uncountable Δ-system $(s_\xi)_{\xi<\omega_1}$ of finite subsets of ω_1 with root s, there is $\xi_0 < \dots < \xi_{n-1} \in \omega_1$ and an $F = \cup F_i \in \mathcal{F}$ so that $s \subset R(F)$, $s_{\xi_i} \setminus s \subset F_i \setminus R(F)$ and

$$\varphi_{F_i F_j}(s_i) = s_j.$$

A construction scheme is capturing if it is n-capturing for all $n < \omega$.

Theorem 4.1.4. *If there is a 3-capturing construction scheme then there is a Suslin-tree.*

In particular, $MA_{\aleph_1}$ is not consistent with capturing construction schemes.

Theorem 4.1.5. *If $\aleph_1$ Cohen reals are added to any model or if $\diamondsuit$ holds then there are capturing construction schemes.*

It is known that CH is not necessarily enough to imply the existence of capturing construction schemes. Finally, we mention that capturing can be used to construct Banach spaces with very tight control over their biorthogonal systems. Let us also remark that capturing construction schemes are related to Velleman's theory of simplified morasses.

4.2 Countable approximations

Solutions to combinatorial problems often follow the same head-on approach: enumerate certain objectives and then inductively meet these goals. Imagine that you are asked to color the points of a topological space with red and blue so that both colors appear on *any* copy of the Cantor-space in X. So, one lists the Cantor-subspaces and inductively declares one point red and one point blue from each; this idea, due to Berstein, works perfectly if X is small i.e. size at most the continuum. However, for larger spaces, we might run into the following problem: after continuum many steps, we could have accidentally covered some Cantor-subspace with red points only. So, how can we avoid such a roadblock?

The methods to meet the goals in the above simple solution scheme vary from problem to problem, however the techniques for *finding the right enumeration* of infinitely or uncountably many objectives frequently involve the same idea. In particular, a recurring feature is to write our set of objectives $\mathcal{X}$ as a union of smaller pieces $\langle \mathcal{X}_\alpha : \alpha < \kappa \rangle$ so that each $\mathcal{X}_\alpha$ resembles the original structure $\mathcal{X}$. This is what we refer to as a *filtration*. In various situations, we need the filtration to consist of countable sets; in others, we require that $\mathcal{X}_\alpha \subseteq \mathcal{X}_\beta$ for $\alpha < \beta < \kappa$. In the modern literature, the sequence $\langle \mathcal{X}_\alpha : \alpha < \kappa \rangle$ is more than often defined by intersecting $\mathcal{X}$ with an increasing chain of countable elementary submodels; in turn, elementarity allows properties of $\mathcal{X}$ to reflect.

The introduction of elementary submodels to solving combinatorial problems was truly revolutionary. It provided deeper insight and simplified proofs to otherwise technical results. Nonetheless, note that any set $\mathcal{X}$ which is covered by an increasing family of countable sets must have size at most $\aleph_1$, a rather serious limitation even when considering problems arising from the reals. Indeed, this is one of the reasons that the assumption $2^{\aleph_0} = \aleph_1$, i.e. the Continuum Hypothesis, is so ubiquitous when dealing with uncountable structures.

On the other hand, several results which seemingly require the use of CH can actually be proved without any extra assumptions. So now the question is, how can we define reasonable filtrations by countable sets to cover structures of size bigger than $\aleph_1$? It turns out that one can relax the assumption of the filtration being increasing in a way which still allows many of our usual arguments for chains to go through. This is done by using a *tree of elementary submodels* rather than chains, an idea which we believe originally appeared in a paper of R. O. Davies [6] in the 1960s.

4.2.1 Trees of countable elementary submodels

The simple idea is that we can always cover a structure of size κ with a continuous chain of elementary submodels of size $< \kappa$ so lets see what happens if we repeat this process and cover each elementary submodel again with chains of smaller submodels, and those submodels with chains of even smaller submodels and so on ... The following result is a simple version of [31, Lemma 3.17]:

Theorem 4.2.1. *Suppose that κ is cardinal, x is a set. Then there is a large enough cardinal θ and a sequence of $\langle M_\alpha : \alpha < \kappa \rangle$ of elementary submodels of $H(\theta)$ so that*

(I) $|M_\alpha| = \omega$ and $x \in M_\alpha$ for all $\alpha < \kappa$,

(II) $\kappa \subset \bigcup_{\alpha<\kappa} M_\alpha$, and

(III) for every $\beta < \kappa$ there is $m_\beta \in \mathbb{N}$ and models $N_{\beta,j} \prec H(\theta)$ such that $x \in N_{\beta,j}$ for $j < m_\beta$ and

$$\bigcup\{M_\alpha : \alpha < \beta\} = \bigcup\{N_{\beta,j} : j < m_\beta\}.$$

We will refer to such a sequence of models as a *Davies-tree for κ over x* in the future (and we will see shortly why they are called trees). The cardinal κ will denote the size of the structures that we deal with (e.g. the size of $\mathbb{R}^2$) while the set x contains the objects relevant to the particular situation (e.g. a set of lines).

Note that if the sequence $\langle M_\alpha : \alpha < \kappa\rangle$ is increasing then $\bigcup\{M_\alpha : \alpha < \beta\}$ is also an elementary submodel of $H(\theta)$ for each $\beta < \kappa$; as we said already, there is no way to cover a set of size bigger than ω_1 with an increasing chain of *countable* sets. Theorem 4.2.1 says that we can cover by countable elementary submodels and almost maintain the property that the initial segments $\bigcup\{M_\alpha : \alpha < \beta\}$ are submodels. Indeed, each initial segment is the union of *finitely many* submodels by condition (3) while these models still contain everything relevant (denoted by x above) as well.

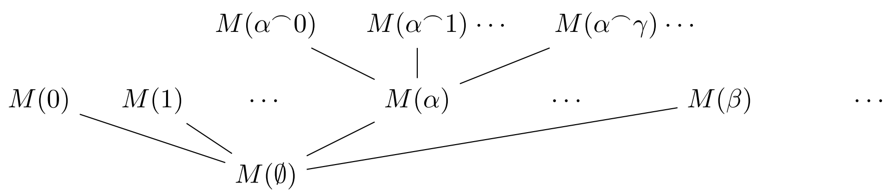
Proof of Theorem 4.2.1. Let θ be large enough so that $\kappa, x \in H(\theta)$. We recursively construct a tree T of finite sequences of ordinals and elementary submodels $M(a)$ for $a \in T$. Let $\emptyset \in T$ and let $M(\emptyset)$ be an elementary submodel of size κ so that

- $x \in M(\emptyset)$,
- $\kappa \subset M(\emptyset)$.

Suppose that we defined a tree T' and corresponding models $M(a)$ for $a \in T'$. Fix $a \in T'$ and suppose that $M(a)$ is *uncountable*. Find a continuous, increasing sequence of elementary submodels $\langle M(a^\frown\xi)\rangle_{\xi<\zeta}$ so that

- $x \in M(a^\frown\xi)$ for all $\xi < \zeta$,
- $M(a^\frown\xi)$ has size strictly less than $M(a)$, and
- $M(a) = \bigcup\{M(a^\frown\xi) : \xi < \zeta\}$.

We extend T' with $\{a^\frown\xi : \xi < \zeta\}$ and iterate this procedure to get T.



It is easy to see that this process produces a downwards closed subtree T of $\mathrm{Ord}^{<\omega}$ and if $a \in T$ is a terminal node then $M(a)$ is countable. Let us well order $\{M(a) : a \in T$ is a terminal node$\}$ by the lexicographical ordering $<_{\mathrm{lex}}$.

First, note that the order type of $<_{\mathrm{lex}}$ is κ since $\{M(a) : a \in T$ is a terminal node$\}$ has size κ and each $M(a)$ has $< \kappa$ many $<_{\mathrm{lex}}$-predecessors.

We wish to show that if $b \in T$ is terminal then $\bigcup\{M(a) : a <_{\text{lex}} b, a \in T$ is a terminal node$\}$ is the union of finitely many submodels containing x. Suppose that $|b| = m \in \mathbb{N}$ and write

$$N_{b,j} = \bigcup\{M((b \restriction j-1)^\frown \xi) : \xi < b(j-1)\}$$

for $j = 1 \dots m$. It is clear that $N_{b,j}$ is an elementary submodel as a union of an increasing chain. Also, if $a <_{\text{lex}} b$ then $M(a) \subset N_{b,j}$ must hold where $j = \min\{i \le n : a(i) \ne b(i)\}$. So

$$\bigcup\{M(a) : a <_{\text{lex}} b \text{ is terminal}\} = \bigcup\{N_{b,j} : j < m\}$$

as desired.

$\square$

Remarks. Note that this proof shows that if $\kappa = \aleph_n$ then every initial segment in the lexicographical ordering is the union of n elementary submodels (the tree T has height n).

In the future, when working with a sequence of elementary submodels $\langle M_\alpha : \alpha < \kappa\rangle$, we use the notation

$$M_{<\beta} = \bigcup\{M_\alpha : \alpha < \beta\}$$

for $\beta < \kappa$.

4.2.2 Paradoxical covers of the plane

Theorem 4.2.2. $\mathbb{R}^2$ *can be covered by countably many rotated graphs of functions.*

Proof. Fix distinct lines ℓ_i for $i < \omega$ through the origin. As before, our goal is to find sets A_i so that $\mathbb{R}^2 = \bigcup\{A_i : i < \omega\}$ and if $\ell \perp \ell_i$ then $|A_i \cap \ell| \le 1$.

Let $\kappa = \mathfrak{c}$ and take a Davies-tree $\langle M_\alpha : \alpha < \kappa\rangle$ for κ over $\{\kappa, \mathbb{R}^2, r, \ell_i : i < \omega\}$ where $r : \kappa \to \mathbb{R}^2$ is onto. So, if $\xi \in \kappa \cap M_\alpha$ then $r(\xi) \in \mathbb{R}^2 \cap M_\alpha$. In turn, $\mathbb{R}^2 \subseteq \bigcup\{M_\alpha : \alpha < \kappa\}$.

By induction on $\beta < \kappa$, we will distribute the points in $\mathbb{R}^2 \cap M_{<\beta}$ among the sets A_i while making sure that if $\ell \perp \ell_i$ then $|A_i \cap \ell| \le 1$. In a general step, we list the countable set $\mathbb{R}^2 \cap M_\beta \setminus M_{<\beta}$ as $\{t_n : n < \omega\}$. Suppose we were able to put t_k into A_{i_k} for $k < n$ and we consider t_n.

Recall that $M_{<\beta}$ can be written as $\bigcup\{N_{\beta,j} : j < m_\beta\}$ for some finite m_β where each $N_{\beta,j}$ is an elementary submodel containing $\{\kappa, \mathbb{R}^2, r, \ell_i : i < \omega\}$. In turn, $\mathbb{R}^2 \cap M_{<\beta}$ is the union of m_β many sets which are closed under constructibility using the lines $\{\ell_i : i < \omega\}$. This means that there could be at most m_β many $i \in \omega \setminus \{i_k : k < n\}$ which is bad for t_n i.e. such i so that the line $\ell(t_n, i)$ through t_n which is perpendicular to ℓ_i already meets A_i. Indeed, otherwise we can find a single $j < m_\beta$ and $i \ne i' \in \omega \setminus \{i_k : k < n\}$ so that the line $\ell(t_n, i)$ meets $A_i \cap N_{\beta,j}$ already and $\ell(t_n, i')$ meets $A_{i'} \cap N_{\beta,j}$ already. However, this means that t_n is constructible from $\mathbb{R}^2 \cap N_{\beta,j}$ so $t_n \in N_{\beta,j}$ as well. This contradicts $t_n \in M_\beta \setminus M_{<\beta}$.

So select any $i_n \in \omega \setminus \{i_k : k < n\}$ which is not bad for t_n and put t_n into A_{i_n}. This finishes the induction and hence the proof of the theorem.

$\square$

The next application, similarly to Davies' result, produces a covering of the plane with small sets. However, this argument makes crucial use of the fact that a set of size $\aleph_m$ (for $m \in \mathbb{N}$) can be covered by a Davies-tree such that the initial segments are expressed as the union of m elementary submodels.

Definition 4.2.3. *We say that $A \subset \mathbb{R}^2$ is a* cloud around a point $a \in \mathbb{R}^2$ *iff every line ℓ through a intersects A in a finite set.*

Note that one or two clouds cannot cover the plane; indeed, if A_i is a cloud around a_i for $i < 2$ then the line ℓ through a_0 and a_1 intersects $A_0 \cup A_1$ in a finite set. How about three or more clouds? The answer comes from a truly surprising result of P. Komjáth and J. H. Schmerl:

Theorem 4.2.4 ([23] and [36])**.** *The following are equivalent for each $m \in \mathbb{N}$:*

1. $2^\omega \le \aleph_m$,
2. $\mathbb{R}^2$ *is covered by at most $m+2$ clouds.*

Moreover, $\mathbb{R}^2$ is always covered by countably many clouds.

We only prove (1) implies (2) and follow Komjáth's original argument for CH. The fact that countably many clouds always cover $\mathbb{R}^2$ can be proved by a simple modification of the proof below.

Proof. Fix $m \in \omega$ and suppose that the continuum is at most $\aleph_m$. In turn, there is a Davies-tree $\langle M_\alpha : \alpha < \aleph_m \rangle$ for $\mathfrak{c}$ over $\mathbb{R}^2$ so that $M_{<\alpha} = \bigcup\{N_{\alpha,j} : j < m\}$ for every $\alpha < \aleph_m$.

Fix $m+2$ points $\{a_k : k < m+2\}$ in $\mathbb{R}^2$ in general position (i.e. no three are collinear). Let $\mathcal{L}^k$ denote the set of lines through a_k and let $\mathcal{L} = \bigcup\{\mathcal{L}^k : k < m+2\}$. We will define clouds A_k around a_k by defining a map $F : \mathcal{L} \to [\mathbb{R}^2]^{<\omega}$ such that $F(\ell) \in [\ell]^{<\omega}$ and letting

$$A_k = \{a_k\} \cup \bigcup\{F(\ell) : \ell \in \mathcal{L}^k\}$$

for $k < m+2$. We have to make sure that for every $x \in \mathbb{R}^2$ there is $\ell \in \mathcal{L}$ so that $x \in F(\ell)$.

Now, let $\mathcal{L}_\alpha = (\mathcal{L} \cap M_\alpha) \setminus M_{<\alpha}$ for $\alpha < \aleph_m$. We define F on L_α for each $\alpha < \aleph_m$ independently so fix an $\alpha < \aleph_m$. List $\mathcal{L}_\alpha \setminus \mathcal{L}'$ as $\{\ell_{\alpha,i} : i < \omega\}$ where $\mathcal{L}'$ is the set of $\binom{m+2}{2}$ lines determined $\{a_k : k < m+2\}$. We let

$$F(\ell_{\alpha,i}) = \bigcup\{\ell \cap \ell_{\alpha,i} : \ell \in \mathcal{L}' \cup \{\ell_{\alpha,i'} : i' < i\}\}$$

for $i < \omega$.

We claim that this definition works: fix a point $x \in \mathbb{R}^2$ and we will show that there is $\ell \in \mathcal{L}$ with $x \in F(\ell)$. Find the unique $\alpha < \aleph_m$ such that $x \in M_\alpha \setminus M_{<\alpha}$. It is easy to see that $\cup\mathcal{L}'$ is covered by our clouds hence we suppose $x \notin \bigcup\mathcal{L}'$. Let ℓ_k denote the line through x and a_k.

Observation 4.2.5. *$|M_{<\alpha} \cap \{\ell_k : k < m+2\}| \le m$.*

Proof. Suppose that this is not true. Then (by the pigeon hole principle) there is $j < m$ such that $|N_{\alpha,j} \cap \{\ell_k : k < m+2\}| \ge 2$ and in particular the intersection of any two of these lines, the point x, is in $N_{\alpha,j} \subset M_{<\alpha}$. This contradicts the minimality of α. □

We achieved that

$$|\{\ell_k : k < m+2\} \cap (\mathcal{L}_\alpha \setminus \mathcal{L}')| \ge 2$$

i.e. there is $i' < i < \omega$ such that $\ell_{\alpha,i'}, \ell_{\alpha,i} \in \{\ell_k : k < m+2\}$. Hence $x \in F(l_{\alpha,j})$ is covered by one of the clouds.

□

4.3 Uncountable approximations

Our main goal in this section is to show that, under certain assumptions, one can a sequence of countably closed elementary submodels, each of size $\mathfrak{c}$, which is reminiscent of Davies-trees while the corresponding models cover structures of size $> \mathfrak{c}^+$; note that this covering would not be possible by an increasing chain of models of size $\mathfrak{c}$.

4.3.1 High Davies-trees

So, what is it exactly that we aim to show? We say that a *high Davies-tree for κ over x* is a sequence $\langle M_\alpha : \alpha < \kappa \rangle$ of elementary submodels of $H(\theta)$ for some large enough regular θ such that

(I) $[M_\alpha]^\omega \subset M_\alpha$, $|M_\alpha| = \mathfrak{c}$ and $x \in M_\alpha$ for all $\alpha < \kappa$,

(II) $[\kappa]^\omega \subset \bigcup_{\alpha<\kappa} M_\alpha$, and

(III) for each $\beta < \kappa$ there are $N_{\beta,j} \prec H(\theta)$ with $[N_{\beta,j}]^\omega \subset N_{\beta,j}$ and $x \in N_{\beta,j}$ for $j < \omega$ such that
$$\bigcup\{M_\alpha : \alpha < \beta\} = \bigcup\{N_{\beta,j} : j < \omega\}.$$

Now, a high Davies-tree is really similar to the Davies-trees we used so far, only that we work with countably closed models of size $\mathfrak{c}$ (instead of countable ones) and the initial segments $M_{<\beta}$ are countable unions of such models (instead of finite unions). Furthermore, we require that the models cover $[\kappa]^\omega$ instead of κ itself. This is because our applications typically require to deal with all countable subsets of a large structure.

One can immediately see that (II) implies that $\kappa^\omega = \kappa$ and so high Davies-trees might not exist for some κ. Nonetheless, a very similar tree-argument to the proof of Theorem 4.2.1 shows that high Davies-trees do exist for the finite successors of $\mathfrak{c}$ i.e. for $\kappa < \mathfrak{c}^{+\omega}$. We will not repeat that proof here but present a significantly stronger result.

As mentioned already, some extra set theoretic assumptions will be necessary to prove the existence of high Davies-trees for cardinals above $\mathfrak{c}^{+\omega}$ so let us recall two notions. We say that $\square_\mu$ holds for a singular μ iff there is a sequence $\langle C_\alpha : \alpha < \mu^+ \rangle$ so that C_α is a closed and unbounded subset of α of size $< \mu$ and $C_\alpha = \alpha \cap C_\beta$ whenever α is an accumulation point of C_β. $\square_\mu$ is known as *Jensen's square principle*; R. Jensen proved that $\square_\mu$ holds for all uncountable μ in the constructible universe L.

Furthermore, a cardinal μ is said to be *ω-inaccessible* iff $\nu^\omega < \mu$ for all $\nu < \mu$. Now, our main theorem is the following:

Theorem 4.3.1. *There is a high Davies-tree $\langle M_\alpha : \alpha < \kappa \rangle$ for κ over x whenever*

1. *$\kappa = \kappa^\omega$, and*

2. *μ is ω-inaccessible, $\mu^\omega = \mu^+$ and $\square_\mu$ holds for all μ with $\mathfrak{c} < \mu < \kappa$ and $\mathrm{cf}(\mu) = \omega$.*

Moreover, the high Davies-tree $\langle M_\alpha : \alpha < \kappa \rangle$ can be constructed so that

3. *$\langle M_\alpha : \alpha < \beta \rangle \in M_\beta$ for all $\beta < \kappa$, and*

4. *$\bigcup\{M_\alpha : \alpha < \kappa\}$ is also a countably closed elementary submodel of $H(\theta)$.*

We will say that $\langle M_\alpha : \alpha < \kappa \rangle$ is a *sage Davies-tree* if it is a high Davies-tree satisfying the extra properties (3) and (4) above. Recall that we have been working with the structure $(H(\theta), \in)$ so far. However, if one wishes to do so, it can be supposed that the models M_α and $N_{\alpha,j}$ are submodels of $(H(\theta), \in, \lhd)$ where $\lhd$ is some well-order on $H(\theta)$. These extra assumptions can be quite useful e.g. the well order $\lhd$ can be used to make *uniform choices in a construction* of say topological spaces and hence the same construction can be reproduced by any model with the appropriate parameters.

Finally, let us remark that if one only aims to construct high Davies-trees (which are not necessary sage) then slightly weaker assumptions than (1) and (2) suffice.

In order to state a rough corollary, recall that (1) and (2) are satisfied by all κ with uncountable cofinality in the constructible universe. Hence:

Corollary 4.3.2. *If $V = L$ then there is a sage Davies-tree for κ over x for any cardinal κ with uncountable cofinality.*

On the other hand, high Davies-trees might not exist at all.

Theorem 4.3.3. *Consistently, relative to a supercompact cardinal, GCH holds and there are no high Davies-trees for any $\kappa \geq \aleph_{\omega+1}$.*

4.3.2 Coloring topological spaces

Our first application concerns a truly classical result due to F. Bernstein: there is a coloring of $\mathbb{R}$ with two colors such that no uncountable Borel set is monochromatic. In other words, the family of Borel sets in $\mathbb{R}$ has chromatic number 2. Indeed, list all the uncountable Borel sets as $\{B_\alpha : \alpha < \mathfrak{c}\}$ and inductively pick distinct $x_\beta, y_\beta \in B_\beta \setminus \{x_\alpha, y_\alpha : \alpha < \beta\}$. This can be done since each B_β contains a Cantor subspace and so has size continuum. Now any map $f : \mathbb{R} \to 2$ that sends $\{x_\alpha : \alpha < \mathfrak{c}\}$ to 0 and $\{y_\alpha : \alpha < \mathfrak{c}\}$ to 1 is as desired.

Now, let $\mathcal{C}(X)$ denote the set of Cantor subspaces of an arbitrary topological space (X, τ). Can we extend Berstein's theorem to general topological spaces? The above simple argument certainly fails if there are more than $\mathfrak{c}$ many Cantor subspaces.

Theorem 4.3.4. *Suppose that (X, τ) is a topological space of size κ. If there is a high Davies-tree for κ over (X, τ) then there is a coloring $f : X \to \mathfrak{c}$ so that $f[C] = \mathfrak{c}$ for any $C \in \mathcal{C}(X)$.*

Hence, if $|X| < \mathfrak{c}^{+\omega}$ or, more generally, κ satisfies the assumptions of Theorem 4.3.1 then such colorings exist.

Proof. Let $\langle M_\alpha : \alpha < \kappa \rangle$ be a high Davies-tree for κ over X. In turn, X and $[X]^\omega$ are covered by $\bigcup\{M_\alpha : \alpha < \kappa\}$. We let $\mathcal{C}_\alpha = \mathcal{C}(X) \cap M_\alpha \setminus M_{<\alpha}$, $X_\alpha = X \cap M_\alpha \setminus M_{<\alpha}$ and $X_{<\alpha} = X \cap M_{<\alpha}$.

Claim 4.3.5. *Suppose that $C \in \mathcal{C}(X)$ and $C \cap X_{<\alpha}$ is uncountable. Then there is a $D \in M_{<\alpha} \cap \mathcal{C}(X)$ such that $D \subseteq C$.*

Proof. Indeed, $M_{<\alpha} = \bigcup\{N_{\alpha,j} : j < \omega\}$ and each $N_{\alpha,j}$ is ω-closed. So there must be an $j < \omega$ such that $C \cap N_{\alpha,j}$ is uncountable. Find $A \subseteq C \cap N_{\alpha,j}$ which is countable and dense in $C \cap N_{\alpha,j}$. Note that A must be an element of $N_{\alpha,j}$ as well and hence, the uncountable closure $\bar{A}$ of A is an element of $N_{\alpha,j}$ (since $\tau \in N_{\alpha,j}$). Now, we can pick $D \subseteq \bar{A} \subseteq C$ such that $D \in N_{\alpha,j} \cap \mathcal{C}(X) \subseteq M_{<\alpha} \cap \mathcal{C}(X)$. □

We define $f_\alpha : X_\alpha \to \mathfrak{c}$ so that $f_\alpha[C] = \mathfrak{c}$ for any $C \in \mathcal{C}_\alpha$ so that $C \cap X_{<\alpha}$ is countable. This can be done just like Berstein's original theorem; indeed let

$$\mathcal{C}^*_\alpha = \{C \cap X_\alpha : C \in \mathcal{C}_\alpha, |C \cap X_{<\alpha}| \leq \omega\}.$$

$\bigcup \mathcal{C}^*_\alpha \subseteq X \cap M_\alpha$ and if $\mathcal{C}^*_\alpha \neq \emptyset$ then $|\bigcup \mathcal{C}^*_\alpha| = \mathfrak{c}$. Moreover, $|\mathcal{C}^*_\alpha| \leq \mathfrak{c}$ and $\mathcal{C}^*_\alpha$ is $\mathfrak{c}$-uniform i.e. each element has size $\mathfrak{c}$. So, we can use the same induction as Berstein to find f_α.

We claim that $f = \bigcup\{f_\alpha : \alpha < \kappa\}$ satisfies the requirements. Indeed, suppose that $C \in \mathcal{C}(X)$ and let α be minimal so that $D \subseteq C$ for some $D \in \mathcal{C}_\alpha$. Claim 4.3.5 implies that $C \cap X_{<\alpha}$ is countable and hence $\mathfrak{c} = f_\alpha[D] \subseteq f[C]$.

□

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