Infinite graph theory II: exercises on 09/06/2022

1. Let $G = (V, E)$ be a graph and let M be an elementary submodel with $|M| \subseteq M$ containing G. Suppose that $u \neq v \in V$ are in the same component of $(V, E \setminus M)$ and F is a uv-cut in this component with $|F| \subseteq |M|$. Show that F separates u and v in G as well. (10p)

   Hint: Suppose indirectly that there is an uv-path P in $G - F$. Show that $\lambda_G(u', v') > |M|$ where u' and v' are the first and the last vertex of P that is in M. Construct a uv-path in $(V, E \setminus (M \cup F))$.

2. The Cycle Double Cover Conjecture says that for every 2-edge-connected graph $G = (V, E)$ there is a family of cycles in G that covers each edge exactly twice. Show that if the conjecture holds for countable graphs then it holds for every graph. (10p)

   Hint: Find a family $G_j = (V_j, E_j)$ $(j \in J)$ of 2-edge-connected countable subgraphs of G such that the sets E_j partition E.

3. Let $G = (A, B; E)$ be a bipartite graph such that for every $X \subseteq A$ the size of its neighbourhood $N(X)$ is at least $|X|$. Show that if G is locally finite, then it admits a matching that covers A but this is not always the case for arbitrary graphs. (10p)

4. Let $G = (V, E)$ be a graph. An augmenting path $P \subseteq E$ for a matching $I \subseteq E$ is a finite path of odd length or a ray in which exactly every second edge belongs to I. Note that $I \triangle P$ is a matching that covers strictly more vertices than I. Construct a graph in which every matching admits an augmenting path but there is no perfect matching.(10p)

   Hint: Use uncountable graphs.