Infinite graph theory II: exercises on 02/06/2022

1. Prove that the cut space is closed under thin sums.

2. Is it true that in any connected graph the fundamental cuts of any ordinary spanning tree generates the cut space in the sense of thin sums?

3. Show that for every graph $G = (V, E)$ and for every infinite cardinal κ, the set E can be partitioned into at most κ subforests of G if and only if V can be linearly ordered in such a way that each $v \in V$ has at most κ down-neighbours (i.e. neighbours in G smaller than v).

4. Let G be a graph and let H be a subgraph of G. Suppose that $F \subseteq E(H)$ is a bond in H but not in G. Show that $G - F$ has a connected component C such that all the end-vertices of the edges in F are in C.

5. Prove that if the graph $G = (V, E)$ does not contain any odd cuts and M is an elementary submodel that contains G, then $(V \cap M, E \cap M)$ is a subgraph of G not containing any odd cuts.

   Hints:

   - If there is an odd cut, then there is an odd bond as well.
   - The finite subsets of M are also elements of M and finite elements of M are also subsets of M (whenever Σ contains a certain finite set of formulas.)
   - The existence of a path P in G between the end-vertices of some fixed $f \in F$ avoiding F can be expressed by a formula with parameters G, F and f.

6. Suppose that $G = (V, E)$ is a graph, $u \neq v \in V$ and M is an elementary submodel with $G, u, v \in M$ and $|M| \subseteq M$. Prove that if u and v are in the same connected component of $(V, E \setminus M)$, then the local edge-connectivity between u and v must be larger than $|M|$.

   Hint: Show that M must contain a minimal uv-cut and it must be also a subset of M.