Infinite graph theory II: exercises on 05/05/2022

1. Suppose that $G = (V, E)$ is a locally finite connected multigraph, $V = \{v_n : n \in \mathbb{N}\}$ is an enumeration, $S_n := \{v_i : i \leq n\}$ and G_n is the finite multigraph that we obtain from G by contracting the components of $G - S_n$ and deleting the arising loops. For $n \in \mathbb{N}$, let T_n be a spanning tree of G_n such that $E(T_n) \subseteq E(T_m)$ for $n \leq m$. Give an example where $F := \bigcup_{n\in\mathbb{N}} E(T_n)$ is not the edge set of a spanning tree of G. Is it true that the standard subspace corresponding to F is always a topological spanning tree of G?

2. Let $G = (V, E)$ be a locally finite connected multigraph. Show that $F \subseteq E$ is a circuit if and only if it is not contained in the edge set of any topological spanning tree of G and minimal with this property.

3. Formulate and prove a characterisation of finite bonds in the spirit of the previous exercise.

4. Let $n \in \mathbb{N}$ with $n \geq 3$ and let $H = (\mathbb{Z}_{n^2} \times [n], F)$ be the graph where

   $$F = \{\{(i, j), (i + 1, k)\} : i \in \mathbb{Z}_{n^2}; j, k \in [n]\}.$$

   Let $\mathcal{P}$ be the set of paths in H of length n and let us fix an enumeration $E(P) = \{e_0, \ldots, e_{n-1}\}$ for each $P \in \mathcal{P}$.

   For a graph $G = (\mathbb{N}, E)$, let $G_P = (\mathbb{N} \times \{P\}, E_P)$ where $\{(k, P), (\ell, P)\} \in E_P$ iff $\{k, \ell\} \in E$. Suppose that graph $G = (\mathbb{N}, E)$ has the following property: If we subdivide each $e \in E(H)$ with new vertices (e, P) for $P \in \mathcal{P}$ with $e \in E(P)$ in an appropriate order and identify the vertex (m, P) of G_P with the vertex (e_m, P) of the subdivided H for $m < n$, then the resulting graph G' is isomorphic to G and the isomorphism can be chosen in such a way that it maps $m \in V(G)$ to $(m \cdot n, 1) \in V(G')$ for every $m < n$.

   Prove that such a graph G exists. Demonstrate that the deletion of the edges of any path between 0 and 1 in such a G results in a disconnected graph which has a component that does not meet $\{0, \ldots, n - 1\}$. Show that the set of the isomorphism classes of such graphs has a smallest element with respect to the subgraph relation and it is k-connected. (10 points)