Infinite graph theory II: exercises on 28/04/2022

1. Let $G = (V, E)$ be a locally finite connected graph and suppose that $F \subseteq E$ contains an edge from each finite cut of G and minimal with respect to this property. Prove that F does not contain any circuits (not even an infinite one).

   Hint: The Jumping Arc Lemma helps

2. Let $G = (V, E)$ be a locally finite connected graph and let $A \subseteq |G|$ be an arc that contains uncountably many ends. Let $<$ be the ordering on $A \cap V$ inherited from $[0, 1]$ through a fixed homeomorphism between $[0, 1]$ and A. Find a subset of $A \cap V$ which is dense in itself with respect to the restriction of $<$.

3. Let $D = (V, E)$ be a digraph and let $X, Y \subseteq V$. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are systems of disjoint XY-paths in D. Show that there is a system $\mathcal{R}$ of disjoint XY-paths such that $V(\mathcal{R}) \cap X \supseteq V(\mathcal{P}) \cap X$ and $V(\mathcal{R}) \cap Y \supseteq V(\mathcal{Q}) \cap Y$.

   Hint: The paths in $\mathcal{R}$ can be chosen in such a way that each $R \in \mathcal{R}$ consists of an initial segment of a path in $\mathcal{P}$ and a terminal segment of a path in $\mathcal{Q}$. Build a bipartite multigraph $G = (\mathcal{P}, \mathcal{Q}, F)$ where F contains one edge between $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ for each common vertex of P and Q. Use the Gale-Shapely theorem with suitable preferences to obtain a stable matching and concatenate the segments of the paths accordingly.

4. Consider the context of the Gale-Shapley theorem and let I_0 and I_1 be stable matchings. Every woman picks the man among her partners in I_0 and in I_1 that she likes better (and picks nobody if she does not have a partner in any of these matchings). Is it true that this always defines a stable matching?