Infinite graph theory II: exercises on 14/04/2022

1. Let $D = (V, E)$ be a digraph, $A, B \subseteq V$, $a \in A$ and suppose that $A - a$ can be linked to B in D (by disjoint directed paths) but the whole A cannot. Show that there is an AB-separation S such that every path-system that links $A - a$ to B must be orthogonal to S. (10 points)

   Hint: Use the infinite version of Menger's theorem together with augmenting paths.

2. Show that for a locally finite G the space $|G|$ is Hausdorff.

3. Let $G = (V, E)$ be a graph. A rooted subtree $T \subseteq G$ is called *normal* if the end-vertices of any T-path (i.e. a path in G with its end-vertices in $V(T)$ but without internal vertices in $V(T)$) are comparable in the tree-order.

   Assume that T is a finite normal tree in the graph G, set C is a component of $G - T$ and x is a maximal vertex in T that has a neighbour in C. Let P be any path that lies in $C + x$ and show that $T \cup P$ is a normal tree with respect to the same root as T. Prove that every connected countable graph admits a normal spanning tree.

4. Suppose that $\omega \in \Omega(G)$ contains k pairwise disjoint rays for every $k \in \mathbb{N}$. Show that ω contains an infinite family of pairwise disjoint rays as well. (10 points)

   Hint: Successive application of (finite) Menger's theorem for a suitable nested sequence of separations.