Infinite graph theory II: exercises on 14/07/2022

Let κ be an uncountable regular cardinal.

1. Show that the intersection of a stationary set and a club in κ is stationary.
2. Prove that $\kappa \setminus \alpha$ is a club for every $\alpha < \kappa$ and hence bounded sets cannot be stationary.
3. Let S be a stationary set in κ. Show that if we partition S into fewer than κ pieces then at least one of them must be stationary.
4. Show that if S is stationary in κ, then either $\{\alpha \in S : \ \mathsf{cf}(\alpha) < \alpha\}$ or the set of the uncountable regular cardinals in S is also stationary.

5. Prove that for any function $f : \kappa \to \kappa$ the set $\{\alpha < \kappa : \ (\forall \beta < \alpha)(f(\beta) < \alpha)\}$ is a club.

   Theorem (Solovay)**.** *Every stationary set in κ can be partitioned into κ many stationary sets.*

   Instructions for the proof. Let a stationary $S \subseteq \kappa$ be given.

   (a) Suppose first that $S' := \{\alpha \in S : \ \mathsf{cf}(\alpha) < \alpha\}$ is stationary. Apply Fodor's lemma with S' and $\alpha \mapsto \mathsf{cf}(\alpha)$ to obtain a stationary S'' and $\lambda < \kappa$ such that $f(\alpha) = \lambda$ for $\alpha' \in S''$. Then $S'' \subseteq S^\kappa_\lambda$ and we are done because this is the special case we have already proved at the lecture.

   (b) Argue that we can assume that S consists of uncountable regular cardinals.

   (c) Show that
   $$S' := \{\alpha \in S : \ S \cap \alpha \text{ is not stationary in } \alpha\}$$
   is stationary in κ. Hint: for a given club C take $\alpha := \min(\mathsf{acc}(C) \cap S)$ and argue that $\alpha \cap \mathsf{acc}(C)$ is a club in α that $S \cap \alpha$ avoids.

   (d) For $\alpha \in S'$, let $\left\langle a^\alpha_\xi : \ \xi < \alpha \right\rangle$ be the increasing enumeration of a club in α that avoids $S' \cap \alpha$. Find an $\eta < \kappa$ such that for every $\beta < \kappa$:
   $$S_\beta := \{\alpha \in S' : \ a^\alpha_\eta > \beta\}$$

is stationary in κ.

Hint: suppose for a contradiction the negation of the statement. Take the diagonal intersection C of the clubs C_η obtained from the indirect assumption. Trim C to a smaller club D by applying exercise 5 with the function $\eta \mapsto \beta_\eta$ (where β_η is a bound coming from the indirect assumption). Pick $\gamma, \alpha \in D \cap S'$ with $\gamma < \alpha$ and show $a^\alpha_\gamma = \gamma$ which is a contradiction since $a^\alpha_\gamma \notin S'$ by definition.

(e) Cut S' into κ many stationary sets by using Fodor's lemma iteratively with the function $\alpha \mapsto a^\alpha_\eta$.

□