Infinite graph theory II: exercises on 07/07/2022

For graphs G and H, let $\mathfrak{MB}(G,H)$ denote the *Maker-Breaker game* where G (more precisely the set of edges) is the board, there are turns (indexed by ordinals) each of which begins with Maker claiming a previously unclaimed edge, after which Breaker does likewise. The game terminates when all the edges are claimed and Maker wins if and only if at the end of the game the subgraph G_M of G induced by the edges claimed by Maker contains a subgraph isomorphic to H.

1. Prove that if Breaker has a winning strategy in $\mathfrak{MB}(G,H)$, then he also has one in every game $\mathfrak{MB}(G',H')$ where G' is a subgraph of G and H' is a supergraph of H.

2. Let $G := H := K_\omega$ and show by "strategy stealing" argument that Breaker cannot have a winning strategy in $\mathfrak{MB}(G,H)$.

3. Let $G := H := K_\omega$ and consider the following Maker-strategy in $\mathfrak{MB}(G,H)$. In a general phase Maker has a finite sequence $v_0,\dots,v_n$ of distinct vertices fixed. Then she picks a vertex v_{n+1} such that Breaker has no edges incident with v_{n+1} and claims the edge v_0v_{n+1}. In a general step of the phase, Maker claims v_iv_{n+1} for the smallest $i \leq n$ for which v_i is connected in Maker's subgraph to all the vertices to which v_{n+1} is connected. If there is no such an i, then the phase ends and a new begins. Show that Maker wins in ω many steps by following this strategy.

4. Suppose that the Continuum Hypothesis is true, i.e. $2^{\aleph_0} = \aleph_1$. Then ω_1 has exactly ω_1 countably infinite subsets. We fix an enumeration $\langle A_\alpha : \alpha < \omega_1\rangle$ of these. We also fix a surjection $e_\alpha : \omega \to \alpha$ for $0 < \alpha < \omega_1$.

   Let $G := K_{\omega_1}$ and $H := K_{\omega_1,\omega}$ and consider the following Breaker-strategy in $\mathfrak{MB}(G,H)$. Whenever Maker plays and edge $\alpha\beta$ with[1] $\alpha > \beta$ and for some $n < \omega$ it is the $(n+1)$th time that Maker plays an edge whose larger endpoint is α, then Breaker plays and edge $\alpha\gamma$ with $\gamma \in e_\alpha(n)$ if it is possible and plays arbitrarily otherwise. Show that this is a winning strategy for Breaker.

[1] We assume that the vertex set of K_{ω_1} is ω_1.